SIGMA-PRIKRY FORCING II: ITERATION SCHEME

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Abstract. In Part I of this series [PRS20], we introduced a class of notions of forcing which we call Σ-Prikry, and showed that many of the known Prikry-type notions of forcing that center around singular cardinals of countable cofinality are Σ-Prikry. We showed that given a Σ-Prikry poset $\mathbb{P}$ and a $\mathbb{P}$-name for a non-reflecting stationary set T, there exists a corresponding Σ-Prikry poset that projects to $\mathbb{P}$ and kills the stationarity of T. In this paper, we develop a general scheme for iterating Σ-Prikry posets and, as an application, we blow up the power of a countable limit of Laver-indestructible supercompact cardinals, and then iteratively kill all non-reflecting stationary subsets of its successor. This yields a model in which the singular cardinal hypothesis fails and simultaneous reflection of finite families of stationary sets holds.

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1. Introduction

In the introduction to Part I of this series [PRS20], we described the need for iteration schemes and the challenges involved in devising such schemes, especially at the level of successor of singular cardinals. The main tool available to obtain consistency results at the level of singular cardinals and their successors is the method of forcing with large cardinals and, in particular, *Prikry-type forcings*. By Prikry-type forcings one usually means to a poset $\mathbb{P} = (P, \leq)$ having the following property.

Prikry Property. *There exists an ordering $\leq^*$ on P coarser than $\leq$ (typically, of a better closure degree) satisfying that for every sentence φ in the forcing language and every $p \in P$ there exists $q \in P$ with $q \leq^* p$ deciding φ.*

In this paper, we develop an iteration scheme for Prikry-type posets, specifically, for the class of Σ-*Prikry forcings* that we introduced in [PRS20] (see Definition 2.3 below). Of course, viable iteration schemes for Prikry-type posets already exists, namely, the Magidor iteration and the Gitik iteration (see [Git10, §6]). In both these cases the ordering $\leq^*$ witnessing the Prikry Property of the iteration can be roughly described as the finite-support iteration of the $\leq^*$-orderings of its components. As the expectation from the final $\leq^*$ is to have an eventually-high closure degree, the two schemes are typically useful in the context where one carries an iteration $\langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha \mid \alpha < \rho \rangle$ with each $\dot{\mathbb{Q}}_\alpha$ being a $\mathbb{P}_\alpha$-name for either a trivial forcing, or a Prikry-type forcing concentrating on the combinatorics of the inaccessible cardinal α. This should be compared with the iteration to control the power function $\alpha \mapsto 2^\alpha$ below some cardinal ρ.

In contrast, in this paper, we are interested in carrying out an iteration of length κ^{++}, where κ is a singular cardinal (or, more generally, forced by the first step of the iteration to become one), and all components of the iteration are Prikry-type forcings that concentrate on the combinatorics of κ or its successor. For this, we will need to allow a support of arbitrarily large size below κ. To be able to lift the Prikry property through an infinite-support iteration, members of the Σ-Prikry class are thus required to possess the following stronger property, which is inspired by the concepts coming from the study of topological Ramsey spaces [Tod10].

Complete Prikry Property. *There is a partition of the ordering $\leq$ into countably many relations $\langle \leq^n \mid n < \omega \rangle$ such that, if we denote $\mathrm{cone}_n(q) := \{r \mid r \leq^n q\}$, then, for every 0-open $U \subseteq P$ (i.e., $q \in U \implies \mathrm{cone}_0(q) \subseteq U$), every $p \in P$ and every $n < \omega$, there exists $q \leq^0 p$ such that $\mathrm{cone}_n(q)$ is either a subset of U or disjoint from U.*

To maintain the above property along the iteration we demand on our posets to satisfy *property* $\mathcal{D}$ (Definition 2.12 below). Succinctly, this property is a game-theoretic abstraction of a standard approach for verifying the Prikry property; it asserts that the "good" player has a winning strategy in a

two-player game in which **I** (the "good" player) works towards *diagonalizing* the sequence of conditions produced by **II** (the "bad" player).

Another parameter that requires attention when devising an iteration scheme is the chain condition of the components to be used. In view of the goal of solving a problem concerning the combinatorics of κ or its successor through an iteration of length κ^{++}, there is a need to know that all counterexamples to our problem will show up at some intermediate stage of the iteration, so that we at least have the chance to kill them all. The standard way to secure the latter is to require that the whole iteration $\mathbb{P}_{\kappa^{++}}$ would have the κ^{++}-chain condition (κ^{++}-cc). As the κ-support iteration of κ^{++}-cc posets need not have the κ^{++}-cc (see [Ros18] for an explicit counterexample), members of the Σ-Prikry class are required to satisfy the following strong form of the κ^{++}-cc:

Linked$_0$ Property. *There exists a map $c : P \to \kappa^+$ satisfying that for all $p, q \in P$, if $c(p) = c(q)$, then p and q are compatible, and, furthermore, $\mathrm{cone}_0(p) \cap \mathrm{cone}_0(q)$ is nonempty.*

In particular, our verification of the chain condition of $\mathbb{P}_{\kappa^{++}}$ will not go through the Δ-system lemma; rather, we will take advantage of a basic fact concerning the density of box products of topological spaces.

Now that we have a way to ensure that all counterexamples show up at intermediate stages, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++}\rangle$, and shall want that, for any $\alpha < \kappa^{++}$, $\mathbb{P}_{\alpha+1}$ will amount to forcing over the model $V^{\mathbb{P}_\alpha}$ to solve a problem suggested by z_α. The standard approach to achieve this is to set $\mathbb{P}_{\alpha+1} := \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset that takes care of z_α. However, the disadvantage of this approach is that if $\mathbb{P}_1$ is a notion of forcing that blows up 2^κ, then any typical poset $\mathbb{Q}_1$ in $V^{\mathbb{P}_1}$ which is designed to add a subset of κ^+ via bounded approximations will fail to have the κ^{++}-cc. To work around this, in our scheme, we set $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$, where $\mathbb{A}(\cdot,\cdot)$ is a functor that, to each Σ-Prikry poset $\mathbb{P}$ and a problem z, produces a Σ-Prikry poset $\mathbb{A}(\mathbb{P}, z)$ that projects onto $\mathbb{P}$ and solves the problem z. A key feature of this functor is that the projection from $\mathbb{A}(\mathbb{P}, z)$ to $\mathbb{P}$ *splits*, that is, in addition to a projection map π from $\mathbb{A}(\mathbb{P}, z)$ onto $\mathbb{P}$, there is a map $\pitchfork$ that goes in the other direction, and the two maps commute in a very strong sense. The exact details may be found in our definition of *forking projection* (see Definition 2.13 below).

A special case of the main result of this paper may be roughly stated as follows.

Main Theorem. *Suppose that $\Sigma = \langle \kappa_n \mid n < \omega\rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal κ. For simplicity, let us say that a notion of forcing $\mathbb{P}$ is* nice *if it has property $\mathcal{D}$, $\mathbb{P} \subseteq H_{\kappa^{++}}$ and $\mathbb{P}$ does not collapse κ^+. Now, suppose that:*

- *$\mathbb{Q}$ is a nice Σ-Prikry notion of forcing;*

- $\mathbb{A}(\cdot,\cdot)$ *is a functor that produces for every nice* Σ*-Prikry notion of forcing* $\mathbb{P}$*, and every* $z \in H_{\kappa^{++}}$*, a corresponding nice* Σ*-Prikry notion of forcing* $\mathbb{A}(\mathbb{P}, z)$*. Moreover,* $\mathbb{A}(\cdot,\cdot)$ *admits a forking projection to* $\mathbb{P}$ *with the weak mixing property;*
- $2^{2^\kappa} = \kappa^{++}$*, so that we may fix a bookkeeping list* $\langle z_\alpha \mid \alpha < \kappa^{++}\rangle$*.*

Then there exists a sequence $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++}\rangle$ *of forcings such that* $\mathbb{P}_1$ *is isomorphic to* $\mathbb{Q}$*,* $\mathbb{P}_{\alpha+1}$ *is isomorphic to* $\mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$*, and, for every pair* $\alpha \leq \beta \leq \kappa^{++}$*,* $\mathbb{P}_\beta$ *projects onto* $\mathbb{P}_\alpha$*. Moreover, if for each nonzero limit ordinal* $\alpha \leq \kappa^{++}$*, a certain canonical subforcing* $\mathring{\mathbb{P}}_\alpha$ *of* $\mathbb{P}_\alpha$ *is dense in* $\mathbb{P}_\alpha$*, then* $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++}\rangle$ *consists of nice* Σ*-Prikry forcings.*

1.1. **Organization of this paper.** In Section 2, we recall the definitions of the Σ-Prikry class, forking projections, and introduce property $\mathcal{D}$ and the weak mixing property.

In Section 3, we present our abstract iteration scheme for Σ-Prikry posets, and prove the Main Theorem of this paper (see Lemmas 3.6 and 3.14).

In Section 4, we present the very first application of our scheme. We carry out an iteration of length κ^{++}, where the first step of the iteration is the *Extender Based Prikry Forcing* (EBPF) due to Gitik and Magidor [GM94, §3] for making $2^\kappa = \kappa^{++}$, and all the later steps are obtained by invoking the functor $\mathbb{A}(\mathbb{P}, z)$ from [PRS20, §6] for killing a nonreflecting stationary set decoded from a $\mathbb{P}$-name z. This functor is due to Sharon [Sha05, §2], and as a corollary, we obtain a correct proof of the main result of [Sha05, §3]:

Corollary. *If* κ *is the limit of a countable increasing sequence of supercompact cardinals, then there exists a cofinality-preserving forcing extension in which* κ *remains a strong limit, every finite collection of stationary subsets of* κ^+ *reflects simultaneously, and* $2^\kappa = \kappa^{++}$*.*

1.2. **Notation and conventions.** Our forcing convention is that $p \leq q$ means that p extends q. We write $\mathbb{P} \downarrow q$ for $\{p \in \mathbb{P} \mid p \leq q\}$. Denote $E^\mu_\theta := \{\alpha < \mu \mid \mathrm{cf}(\alpha) = \theta\}$. The sets $E^\mu_{<\theta}$ and $E^\mu_{>\theta}$ are defined in a similar fashion. For a stationary subset S of a regular uncountable cardinal μ, we write $\mathrm{Tr}(S) := \{\gamma \in E^\mu_{>\omega} \mid S \cap \gamma \text{ is stationary in } \gamma\}$. H_ν denotes the collection of all sets of hereditary cardinality less than ν. For every set of ordinals x, we denote $\mathrm{cl}(x) := \{\sup(x \cap \gamma) \mid \gamma \in \mathrm{Ord}, x \cap \gamma \neq \emptyset\}$, $\mathrm{acc}(x) := \{\gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0\}$ and $\mathrm{nacc}(x) := x \setminus \mathrm{acc}(x)$.

2. The Σ-Prikry class and forking projections

In this section, we recall some definitions and facts from [PRS20, §2] and [PRS20, §4], and then continue developing the theory of *forking projections*. Familiarity with [PRS20] is not assumed here.

2.1. The Σ-Prikry class and Property $\mathcal{D}$.

Definition 2.1. We say that $(\mathbb{P},\ell)$ is a *graded poset* iff $\mathbb{P}=(P,\leq)$ is a poset, $\ell:P\to\omega$ is a surjection, and, for all $p\in P$:

- For every $q\leq p$, $\ell(q)\geq\ell(p)$;
- There exists $q\leq p$ with $\ell(q)=\ell(p)+1$.

Convention 2.2. For a graded poset as above, we denote $P_n:=\{p\in P\mid \ell(p)=n\}$, $P^p_n:=\{q\in P\mid q\leq p,\ell(q)=\ell(p)+n\}$, and sometimes write $q\leq^n p$ (and say the q is *an n-step extension* of p) rather than writing $q\in P^p_n$.

Definition 2.3. Suppose that $\mathbb{P}=(P,\leq)$ is a notion of forcing with a greatest element $\mathbb{1}$, and that $\Sigma=\langle\kappa_n\mid n<\omega\rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ. Suppose that μ is a cardinal such that $\mathbb{1}\Vdash_{\mathbb{P}}\check{\mu}=\kappa^+$. For functions $\ell:P\to\omega$ and $c:P\to\mu$, we say that $(\mathbb{P},\ell,c)$ is Σ-*Prikry* iff all of the following hold:

(1) $(\mathbb{P},\ell)$ is a graded poset;
(2) For all $n<\omega$, $\mathbb{P}_n:=(P_n\cup\{\mathbb{1}\},\leq)$ contains a dense subposet $\mathring{\mathbb{P}}_n$ which is κ_n-directed-closed;
(3) For all $p,q\in P$, if $c(p)=c(q)$, then $P^p_0\cap P^q_0$ is non-empty;
(4) For all $p\in P$, $n,m<\omega$ and $q\leq^{n+m}p$, the set $\{r\leq^n p\mid q\leq^m r\}$ contains a greatest element which we denote by $m(p,q)$.[1] In the special case $m=0$, we shall write $w(p,q)$ rather than $0(p,q)$;[2]
(5) For all $p\in P$, the set $W(p):=\{w(p,q)\mid q\leq p\}$ has size $<\mu$;
(6) For all $p'\leq p$ in P, $q\mapsto w(p,q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
(7) Suppose that $U\subseteq P$ is a 0-open set, i.e., $r\in U$ iff $P^r_0\subseteq U$. Then, for all $p\in P$ and $n<\omega$, there is $q\leq^0 p$, such that, either $P^q_n\cap U=\emptyset$ or $P^q_n\subseteq U$.

Remark 2.4. (i) Clause (2) differs from that of [PRS20, Definition 2.3], where we originally required $\mathbb{P}_n$ itself to be κ_n-directed-closed.
(ii) Clause (3) is the Introduction's *Linked$_0$ property*. Often, we will want to avoid encodings and opt to define the function c as a map from P to some natural set $\mathfrak{M}$ of size $\leq\mu$, instead of a map to the cardinal μ itself. In the special case that $\mu^{<\mu}=\mu$, we shall simply take $\mathfrak{M}$ to be H_μ.
(iii) Clause (7) is the *Complete Prikry Property* (CPP).

Definition 2.5. Let $p\in P$. For each $n<\omega$, we write $W_n(p):=\{w(p,q)\mid q\in P^p_n\}$, and $W_{\geq n}(p):=\{w(p,q)\mid\exists m\in\omega\setminus n[q\in P^p_m]\}$. The object $W(p):=\bigcup_{n<\omega}W_n(p)$ is called *the p-tree*.

Fact 2.6 ([PRS20, Lemma 2.8])**.** *Let $p\in P$.*

(1) For every $n<\omega$, $W_n(p)$ is a maximal antichain in $\mathbb{P}\downarrow p$;
(2) Every two compatible elements of $W(p)$ are comparable;

[1] By convention, a greatest element, if exists, is unique.
[2] Note that $w(p,q)$ is the weakest n-step extension of p above q.

(3) For any pair $q' \leq q$ in $W(p)$, $q' \in W(q)$;
(4) $c \restriction W(p)$ is injective.

Fact 2.7 ([PRS20, Lemma 2.10])**.**

(1) $\mathbb{P}$ does not add bounded subsets of κ;
(2) For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \Vdash_{\mathbb{P}} \mathrm{cf}(\nu) < \kappa$, then there exists $p' \leq p$ with $|W(p')| \geq \nu$.[3]

Definition 2.8. We say that $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a *good enumeration* of a set A iff $\vec{r}$ is injective, χ is a cardinal, and $\{r_\xi \mid \xi < \chi\} = A$.

Definition 2.9 (Diagonalizability)**.** Given $p \in P$, $n < \omega$, and a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, we say that $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is *diagonalizable* (with respect to $\vec{r}$) iff the two hold:

(a) $q_\xi \leq^0 r_\xi$ for every $\xi < \chi$;
(b) there is $p' \leq^0 p$ such that for every $q' \in W_n(p')$, $q' \leq^0 q_\xi$, where ξ is the unique index to satisfy $r_\xi = w(p, q')$.

Definition 2.10 (Diagonalizability game)**.** Given $p \in P$, $n < \omega$, a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, and a dense subset D of $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, $\Game_{\mathbb{P}}(p, \vec{r}, D)$ is a game of length χ between two players **I** and **II**, defined as follows:

- At stage $\xi < \chi$, **I** plays a condition $p_\xi \leq^0 p$ compatible with r_ξ, and then **II** plays $q_\xi \in D$ such that $q_\xi \leq p_\xi$ and $q_\xi \leq^0 r_\xi$;
- **I** wins the game iff the resulting sequence $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable.

In the special case that D is all of $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, we omit it, writing $\Game_{\mathbb{P}}(p, \vec{r})$.

The following lemma will be useful later.

Lemma 2.11. *Given $p \in P$, $n < \omega$, a good enumeration $\vec{r}$ of $W_n(p)$, and a dense subset D of $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, **I** has a winning strategy for $\Game_{\mathbb{P}}(p, \vec{r}, D)$ iff it has a winning strategy for $\Game_{\mathbb{P}}(p, \vec{r})$.*

Proof. Only the forward implication requires an argument. Write $\vec{r}$ as $\langle r_\xi \mid \xi < \chi \rangle$; we shall describe a winning strategy for **I** in the game $\Game_{\mathbb{P}}(p, \vec{r})$ by producing sequences of the form $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$, where $\langle (p_\eta, q_\eta) \mid \eta < \xi \rangle$ is an initial play (consisting of ξ rounds) in the game $\Game_{\mathbb{P}}(p, \vec{r})$, and $\langle (p_\eta, q'_\eta) \mid \eta < \xi \rangle$ is an initial play in the game $\Game_{\mathbb{P}}(p, \vec{r}, D)$.

Assuming that **I** has a winning strategy for $\Game_{\mathbb{P}}(p, \vec{r}, D)$, here is a description of our winning strategy for **I** in the game $\Game_{\mathbb{P}}(p, \vec{r})$:

▶ For $\xi = 0$, we play a condition p_0 according to the winning strategy of **I** in the game $\Game_{\mathbb{P}}(p, \vec{r}, D)$. Then, **II** plays $q_0 \leq p_0$ such that $q_0 \leq^0 r_0$. Since D is dense in $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, we then pick $q'_0 \in D$ with $q'_0 \leq^0 q_0$.

▶ Suppose that $\xi < \chi$ is nonzero and that $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$ has already been defined. Let p_ξ be given by the winning strategy of **I** for the

[3]For future reference, we point out that this fact relies only on Clauses (1), (2), (4) and (7) of Definition 2.3. Furthermore, we do not need to know that $\mathbf{1}$ decides a value for κ^+.

game $\Game_{\mathbb{P}}(p,\vec{r},D)$ with respect to the initial play $\langle (p_\eta, q'_\eta) \mid \eta<\xi\rangle$. Then, **II** plays $q_\xi \leq p_\xi$ such that $q_\xi \leq^0 r_\xi$. Finally, pick $q'_\xi \in D$ such that $q'_\xi \leq^0 q_\xi$.

At the end of the above process, since $\langle (p_\xi, q'_\xi) \mid \xi<\chi\rangle$ is a play in the game $\Game_{\mathbb{P}}(p,\vec{r},D)$ using the winning strategy of **I**, we may fix $p' \leq^0 p$ witnessing that $\langle q'_\xi \mid \xi<\chi\rangle$ is diagonalizable. So, for every $q' \in W_n(p')$, if ξ is the unique index to satisfy $r_\xi = w(p,q')$, then $q' \leq^0 q'_\xi \leq^0 q_\xi$. In particular, p' witnesses that $\langle q_\xi \mid \xi<\chi\rangle$ is diagonalizable, as desired. □

Definition 2.12 (Property $\mathcal{D}$)**.** We say that $(\mathbb{P},\ell_{\mathbb{P}})$ has *property* $\mathcal{D}$ iff for any $p\in P$, $n<\omega$ and any good enumeration $\vec{r}=\langle r_\xi \mid \xi<\chi\rangle$ of $W_n(p)$, **I** has a winning strategy for the game $\Game_{\mathbb{P}}(p,\vec{r})$.

2.2. **Forking projections.** In this and the next subsection, we continue the work started in [PRS20, §4] concerning forking projections. This will play a key role in Section 3, where we deal with iterating Σ-Prikry posets.

Definition 2.13 ([PRS20, Definition 4.1])**.** Suppose that $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$ is a Σ-Prikry triple, $\mathbb{A}=(A,\trianglelefteq)$ is a notion of forcing, and $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\mathrm{dom}(\ell_{\mathbb{A}})=\mathrm{dom}(c_{\mathbb{A}})=A$.

A pair of functions $(\pitchfork,\pi)$ is said to be a *forking projection* from $(\mathbb{A},\ell_{\mathbb{A}})$ to $(\mathbb{P},\ell_{\mathbb{P}})$ iff all of the following hold:

(1) π is a projection from $\mathbb{A}$ onto $\mathbb{P}$, and $\ell_{\mathbb{A}}=\ell_{\mathbb{P}}\circ\pi$;
(2) for all $a\in A$, $\pitchfork(a)$ is an order-preserving function from $(\mathbb{P}\downarrow\pi(a),\leq)$ to $(\mathbb{A}\downarrow a,\trianglelefteq)$;
(3) for all $p\in P$, $\{a\in A\mid \pi(a)=p\}$ admits a greatest element, which we denote by $\lceil p\rceil^{\mathbb{A}}$;
(4) for all $n,m<\omega$ and $b\trianglelefteq^{n+m} a$, $m(a,b)$ exists and satisfies:
$$m(a,b)=\pitchfork(a)(m(\pi(a),\pi(b)));$$
(5) for all $a\in A$ and $q\leq\pi(a)$, $\pi(\pitchfork(a)(q))=q$;
(6) for all $a\in A$ and $q\leq\pi(a)$, $a=\lceil\pi(a)\rceil^{\mathbb{A}}$ iff $\pitchfork(a)(q)=\lceil q\rceil^{\mathbb{A}}$;
(7) for all $a\in A$, $a'\trianglelefteq^0 a$ and $r\leq^0\pi(a')$, $\pitchfork(a')(r)\trianglelefteq\pitchfork(a)(r)$.

The pair $(\pitchfork,\pi)$ is said to be a forking projection from $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ to $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$ iff, in addition to all of the above, the following holds:

(8) for all $a,a'\in A$, if $c_{\mathbb{A}}(a)=c_{\mathbb{A}}(a')$, then $c_{\mathbb{P}}(\pi(a))=c_{\mathbb{P}}(\pi(a'))$ and, for all $r\in P_0^{\pi(a)}\cap P_0^{\pi(a')}$, $\pitchfork(a)(r)=\pitchfork(a')(r)$.

Remark 2.14. Intuitively speaking, $\pitchfork(a)$ is an operator that, for each condition $p\in P\downarrow\pi(a)$, provides the $\trianglelefteq$-greatest condition $b\trianglelefteq a$ with $\pi(b)=p$.

Example 2.15. Suppose that $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$ is a Σ-Prikry triple. Let μ denote the cardinal such that $\mathbb{1}\Vdash_{\mathbb{P}}\check{\mu}=\kappa^+$. We define the following objects:

- $\mathbb{A}=(A,\trianglelefteq)$, where $A:=P\times\mu$ and $(p,\alpha)\trianglelefteq(q,\beta)$ iff $p\leq q$ and $\alpha\supseteq\beta$;
- $\ell_{\mathbb{A}}:A\to\omega$ via $\ell_{\mathbb{A}}(p,\alpha):=\ell_{\mathbb{P}}(p)$;
- $c_{\mathbb{A}}:A\to\mu\times\mu$ via $c_{\mathbb{A}}(p,\alpha):=(c_{\mathbb{P}}(p),\alpha)$;
- $\pi:A\to P$ via $\pi(p,\alpha):=p$;

- for $a=(p,\alpha)\in A$, define $\Cap(a):\mathbb{P}\downarrow p\to A$ via $\Cap(a)(q):=(q,\alpha)$.

Then $(\Cap,\pi)$ is a forking projection from $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ to $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$, and $\lceil a\rceil^{\mathbb{A}}=(\pi(a),0)$ for all $a\in A$.

Definition 2.16. Given two posets $\mathbb{P}=(P,\leq)$ and $\mathbb{A}=(A,\trianglelefteq)$, and a projection π from $\mathbb{A}$ to $\mathbb{P}$, we denote by $\mathbb{A}^{\pi}$ the poset $(A,\trianglelefteq^{\pi})$, where $a\trianglelefteq^{\pi}b$ iff $a\trianglelefteq b$ and $\pi(a)=\pi(b)$.

For a subposet $\mathring{\mathbb{A}}=(\mathring{A},\trianglelefteq)$ of $\mathbb{A}$, we likewise denote $\mathring{\mathbb{A}}^{\pi}:=(\mathring{A},\trianglelefteq^{\pi})$.

Lemma 2.17. *Suppose that $(\Cap,\pi)$ is a forking projection from $(\mathbb{A},\ell_{\mathbb{A}})$ to $(\mathbb{P},\ell_{\mathbb{P}})$. For every $a\in A$, $\Cap(a)(\pi(a))=a$.*

Proof. By Definition 2.13(4), using $(n,m,b):=(0,0,a)$, we infer that

$$\Cap(a)(\pi(a))=\Cap(a)(w(\pi(a),\pi(a)))=w(a,a)=a.\qquad\square$$

Lemma 2.18 (Canonical form)**.** *Suppose that $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$ and $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ are both Σ-Prikry notions of forcing. Denote $\mathbb{P}=(P,\leq)$ and $\mathbb{A}=(A,\trianglelefteq)$.*

If $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}})$ admits a forking projection *to $(\mathbb{P},\ell_{\mathbb{P}},c_{\mathbb{P}})$ as witnessed by a pair $(\Cap,\pi)$, then we may assume that all of the following hold true:*

(1) each element of A is a pair (x,y) with $\pi(x,y)=x$;
(2) for all $a\in A$, $\lceil\pi(a)\rceil^{\mathbb{A}}=(\pi(a),\emptyset)$;
(3) for all $p,q\in P$, if $c_{\mathbb{P}}(p)=c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p\rceil^{\mathbb{A}})=c_{\mathbb{A}}(\lceil q\rceil^{\mathbb{A}})$.

Proof. By applying a bijection, we may assume that $A=|A|$ with $\mathbb{1}_{\mathbb{A}}=\emptyset$. To clarify what we are about to do, we agree to say that "a is a lift" iff $a=\lceil\pi(a)\rceil^{\mathbb{A}}$. Now, define $f:A\to P\times A$ via:

$$f(a):=\begin{cases}(\pi(a),\emptyset), & \text{if } a \text{ is a lift;}\\ (\pi(a),a), & \text{otherwise.}\end{cases}$$

Claim 2.18.1. *f is injective.*

Proof. Suppose $a,a'\in A$ with $f(a)=f(a')$.

▶ If a is not a lift and a' is not a lift, then from $f(a)=f(a')$ we immediately get that $a=a'$.

▶ If a is a lift and a' is a lift, then from $f(a)=f(a')$, we infer that $\pi(a)=\pi(a')$, so that $a=\lceil\pi(a)\rceil^{\mathbb{A}}=\lceil\pi(a')\rceil^{\mathbb{A}}=a'$.

▶ If a is not a lift, but a' is a lift, then from $f(a)=f(a')$, we infer that $a=\emptyset=\mathbb{1}_{\mathbb{A}}$, contradicting the fact that $\mathbb{1}_{\mathbb{A}}=\lceil\mathbb{1}_{\mathbb{P}}\rceil^{\mathbb{A}}=\lceil\pi(\mathbb{1}_{\mathbb{A}})\rceil^{\mathbb{A}}$ is a lift. So this case is void. $\square$

Let $B:=\mathrm{Im}(f)$ and $\trianglelefteq_B:=\{(f(a),f(b))\mid a\trianglelefteq b\}$, so that $\mathbb{B}:=(B,\trianglelefteq_B)$ is isomorphic to $\mathbb{A}$. Define $\ell_{\mathbb{B}}:=\ell_{\mathbb{A}}\circ f^{-1}$ and $\pi_{\mathbb{B}}:=\pi\circ f^{-1}$. Also, define $\Cap_{\mathbb{B}}$ via $\Cap_{\mathbb{B}}(b)(p):=f(\Cap(f^{-1}(b))(p))$. It is clear that $b\in B$ is a lift iff $f^{-1}(a)$ is a lift iff $b=(\pi_{\mathbb{B}}(b),\emptyset)$.

Next, define $c_{\mathbb{B}}:B\to\mu\times 2$ by letting for all $b\in B$:

$$c_{\mathbb{B}}(b):=\begin{cases}(c_{\mathbb{P}}(\pi_{\mathbb{B}}(b)),0), & \text{if } b \text{ is a lift;}\\ (c_{\mathbb{A}}(f^{-1}(b)),1), & \text{otherwise.}\end{cases}$$

Claim 2.18.2. *Suppose $b_0, b_1 \in B$ with $c_{\mathbb{B}}(b_0) = c_{\mathbb{B}}(b_1)$. Then $c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_0)) = c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_1))$ and, for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\Cap_{\mathbb{B}}(b_0)(r) = \Cap_{\mathbb{B}}(b_1)(r)$.*

Proof. We focus on verifying that for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\Cap_{\mathbb{B}}(b_0)(r) = \Cap_{\mathbb{B}}(b_1)(r)$. For each $i < 2$, denote $a_i := f^{-1}(b_i)$ and $p_i := \pi_{\mathbb{B}}(b_i)$, so that $\pi(a_i) = p_i$. Suppose $r \in P_0^{p_0} \cap P_0^{p_1}$.

▶ If b_0 is a lift, then so are b_1, a_0, a_1. Therefore, for each $i < 2$, Definition 2.13(6) implies that $\Cap_{\mathbb{B}}(b_i)(r) = f(\Cap(a_i)(r)) = f(\lceil r \rceil^{\mathbb{A}}) = \lceil r \rceil^{\mathbb{B}}$. In effect, $\Cap_{\mathbb{B}}(b_0)(r) = \Cap_{\mathbb{B}}(b_1)(r)$, as desired.

▶ Otherwise, $c_{\mathbb{A}}(a_0) = c_{\mathbb{A}}(a_1)$. As $r \in P_0^{\pi(a_0)} \cap P_0^{\pi(a_1)}$, $\Cap_{\mathbb{B}}(b_0)(p) = f(\Cap(a_0)(p)) = f(\Cap(a_1)(p)) = \Cap_{\mathbb{B}}(b_1)(p)$. □

This completes the proof. □

Setup 2. Throughout the rest of this section, suppose that:

- $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{P}}$;
- $\mathbb{A} = (A, \trianglelefteq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{A}}$;
- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ, and μ is a cardinal such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$;
- $\ell_{\mathbb{P}}$ and $c_{\mathbb{P}}$ are functions witnessing that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is Σ-Prikry;
- $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\mathrm{dom}(\ell_{\mathbb{A}}) = \mathrm{dom}(c_{\mathbb{A}}) = A$;
- $(\Cap, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$.

The next two facts will help verifying Clauses (1) and (3) of Definition 2.3 for the different stages of the iteration in Section 3.

Fact 2.19 ([PRS20, Lemma 4.3])**.** *Suppose that $(\Cap, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$, or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 2.13. For each $a \in A$, the following holds:*

(1) $\Cap(a) \restriction W(\pi(a))$ forms a bijection from $W(\pi(a))$ to $W(a)$;

(2) for all $n < \omega$ and $r \in P_n^{\pi(a)}$, $\Cap(a)(r) \in A_n^a$.

In particular, $(\mathbb{A}, \ell_{\mathbb{A}})$ is a graded poset.

Fact 2.20 ([PRS20, Lemma 4.7])**.** *Suppose that $(\Cap, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, or, just a pair of maps satisfying Clauses (1), (2), (4), (7) and (8) of Definition 2.13. For all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, then $A_0^a \cap A_0^{a'}$ is non-empty. In particular, if $|\mathrm{Im}(c_{\mathbb{A}})| \leq \mu$, then $(\mathbb{A}, \ell_{\mathbb{A}})$ is μ^+-2-linked$_0$.*

Lemma 2.21. *Suppose that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$. Then it has the* CPP.

Proof. Let $U \subseteq A$ be a 0-open set, $a \in A$ and $n < \omega$. Let $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$. Let $\langle (a_\xi, b_\xi) \mid \xi < \chi \rangle$ list the rounds of the game $\Game_{\mathbb{A}}(a, \vec{r})$ in which, in round ξ, **I** plays according to their winning strategy and **II** plays $b_\xi \trianglelefteq^n a_\xi$ such that

(i) $b_\xi \trianglelefteq^0 r_\xi$, and

(ii) if $A_0^{r_\xi} \cap U \neq \emptyset$, then $b_\xi \in U$.

Let $a' \trianglelefteq^0 a$ be a condition witnessing the diagonalizability of $\langle b_\xi \mid \xi < \chi\rangle$. Set $p := \pi(a)$ and $p' := \pi(a')$. By Fact 2.19, $W(a) = \Cap(a)$"$W(p)$, hence, for each $q \leq^n p$, we may let $\xi(q) < \chi$ be such that $\Cap(a)(w(p,q)) = r_{\xi(q)}$. Set $\bar{U} := \{q \in P_n^p \mid b_{\xi(q)} \in U\}$. As $q' \leq^0 q \leq^n p$ implies $\xi(q) = \xi(q')$, the set $\bar{U}$ is 0-open. Recalling Setup 2, $(\mathbb{P}, \ell_{\mathbb{P}}, c)$ is Σ-Prikry, so applying CPP to $\bar{U}$, p', and n, we find $\bar{p} \leq^0 p'$ such that either $P_n^{\bar{p}} \subseteq \bar{U}$ or $P_n^{\bar{p}} \cap \bar{U} = \emptyset$.

Set $\bar{a} := \Cap(a')(\bar{p})$. Since $\bar{p} \leq^0 p' \leq^0 p$, Clauses (1) and (2) of Definition 2.13 yield $\bar{a} \trianglelefteq^0 a' \trianglelefteq^0 a$.

Claim 2.21.1. *Let $b \in A_n^{\bar{a}}$. Then:*

(1) $b \trianglelefteq^0 b_{\xi(\pi(b))}$;

(2) If $b \in U$, then $P_n^{\bar{p}} \subseteq \bar{U}$.

Proof. Denote $q := \pi(b)$.

(1) Since $w(a', b) \in W_n(a')$ and a' is a witness to diagonalizability of $\langle b_\xi \mid \xi < \chi\rangle$, $b \trianglelefteq^0 w(a', b) \trianglelefteq^0 b_\xi$, where ξ is the unique index to satisfy $r_\xi = w(a, b)$. By Clause (4) of Definition 2.13,

$$r_\xi = w(a, b) = \Cap(a)(w(p, q)) = r_{\xi(q)},$$

so that $\xi = \xi(\pi(b))$.

(2) Assuming that $b \in U$, we altogether infer that $b \in A_0^{r_{\xi(q)}} \cap U \neq \emptyset$, and then Clause (ii) above implies that $b_{\xi(q)} \in U$. By the definition of $\bar{U}$, then, $q \in \bar{U} \cap P_n^{\bar{p}}$. So, by the choice of $\bar{p}$, furthermore $P_n^{\bar{p}} \subseteq \bar{U}$. □

It thus follows that if $A_n^{\bar{a}} \cap U \neq \emptyset$, then for every $b \in A_n^{\bar{a}}$, $\pi(b) \in P_n^{\bar{p}} \subseteq \bar{U}$, so that $b_{\xi(\pi(b))} \in U$. By the preceding claim, $b \trianglelefteq^0 b_{\xi(\pi(b))}$, so, since U is 0-open, $b \in U$. Thus we have shown that if $A_n^{\bar{a}} \cap U \neq \emptyset$, then $A_n^{\bar{a}} \subseteq U$. □

Proposition 2.22. *Let $a \in A$, $n < \omega$ and $\vec{s} = \langle s_\xi \mid \xi < \chi\rangle$ be a good enumeration of $W_n(a)$. Let $p' \leq^0 \pi(a)$.*

Suppose that $\langle b_\xi \mid \xi < \chi\rangle$ is a sequence of conditions in $\mathbb{A} \downarrow a$ such that:

(α) $\langle \pi(b_\xi) \mid \xi < \chi\rangle$ is diagonalizable with respect to $\langle \pi(s_\xi) \mid \xi < \chi\rangle$, as witnessed by p';[4]

(β) b is a condition in $\mathbb{A}$ with $\pi(b) = p'$ such that, for all $q' \in W_n(p')$,

$$\Cap(b)(q') \trianglelefteq^0 b_\xi,$$

where ξ is the unique index such that $\pi(s_\xi) = w(\pi(a), q')$.

Then b witnesses that $\langle b_\xi \mid \xi < \chi\rangle$ is diagonalizable with respect to $\vec{s}$.

Proof. We go over the two clauses of Definition 2.9:

(a) Let $\xi < \chi$. By Clause (α) above, $\pi(b_\xi) \leq^0 \pi(s_\xi)$. Together with Definition 2.3(6), it follows that

$$w(\pi(a), \pi(b_\xi)) \leq^0 w(\pi(a), \pi(s_\xi)) = \pi(s_\xi).$$

[4]By Fact 2.19, $\langle \pi(s_\xi) \mid \xi < \chi\rangle$ is a good enumeration of $W_n(\pi(a))$.

Finally, Clauses (1), (4) and (5) of Definition 2.13 yield

$$b_\xi \trianglelefteq^0 w(a,b_\xi) = \Cap(a)(w(\pi(a),\pi(b_\xi))) \trianglelefteq^0 \Cap(a)(\pi(s_\xi)) = s_\xi.$$

(b) Let $b' \in W_n(b)$, and we shall show that $b' \trianglelefteq^0 b_\xi$, where ξ is the unique index to satisfy $s_\xi = w(a,b')$. Set $q' := \pi(b')$. As $\pi(b) = p'$, we infer from Definition 2.13(4) that $b' = \Cap(b)(q')$ and $q' \in W_n(p')$. Thus, by Clause (β) above $b' = \Cap(b)(q') \trianglelefteq^0 b_\xi$, where ξ is the unique index such that $\pi(s_\xi) = w(\pi(a),q')$. Again by Definition 2.13(4),

$$s_\xi = \Cap(a)(\pi(s_\xi)) = \Cap(a)(w(\pi(a),q')) = w(a,b'),$$

as desired. □

2.3. **Types and the Weak Mixing Property.** In this subsection, we will provide a sufficient condition for $(\mathbb{A},\ell_\mathbb{A})$ to inherit property $\mathcal{D}$ from $(\mathbb{P},\ell_\mathbb{P})$.

While reading the next two definitions, the reader may want to have a simple example in mind; such an example is given by Lemma 2.26 below.

Definition 2.23 (Types)**.** A *type over* $(\Cap,\pi)$ is a map $\mathtt{tp}\colon A \to {}^{<\mu}\omega$ having the following properties:

(1) for each $a \in A$, either $\mathrm{dom}(\mathtt{tp}(a)) = \alpha+1$ for some $\alpha<\mu$, in which case we define $\mathrm{mtp}(a) := \mathtt{tp}(a)(\alpha)$, or $\mathtt{tp}(a)$ is empty, in which case we define $\mathrm{mtp}(a) := 0$;
(2) for all $a,b \in A$ with $b \trianglelefteq a$, $\mathrm{dom}(\mathtt{tp}(a)) \leq \mathrm{dom}(\mathtt{tp}(b))$ and for each $i \in \mathrm{dom}(\mathtt{tp}(a))$, $\mathtt{tp}(b)(i) \leq \mathtt{tp}(a)(i)$;
(3) for all $a \in A$ and $q \leq \pi(a)$, $\mathrm{dom}(\mathtt{tp}(\Cap(a)(q))) = \mathrm{dom}(\mathtt{tp}(a))$;
(4) for all $a \in A$, $\mathtt{tp}(a) = \emptyset$ iff $a = \lceil \pi(a) \rceil^\mathbb{A}$;
(5) for all $a \in A$ and $\alpha \in \mu \setminus \mathrm{dom}(\mathtt{tp}(a))$, there exists a *stretch of a to α*, denoted $a^\frown\alpha$, and satisfying the following:
  (a) $a^\frown\alpha \trianglelefteq^\pi a$;
  (b) $\mathrm{dom}(\mathtt{tp}(a^\frown\alpha)) = \alpha+1$;
  (c) $\mathtt{tp}(a^\frown\alpha)(i) \leq \mathrm{mtp}(a)$ whenever $\mathrm{dom}(\mathtt{tp}(a)) \leq i \leq \alpha$;
(6) for all $a,b \in A$ with $\mathrm{dom}(\mathtt{tp}(a)) = \mathrm{dom}(\mathtt{tp}(b))$, for every $\alpha \in \mu \setminus \mathrm{dom}(\mathtt{tp}(a))$, if $b \trianglelefteq a$, then $b^\frown\alpha \trianglelefteq a^\frown\alpha$;
(7) For each $n<\omega$, the poset $\mathring{\mathbb{A}}_n$ is dense in $\mathbb{A}_n$, where $\mathring{\mathbb{A}}_n := (\mathring{A}_n, \trianglelefteq)$ and $\mathring{A}_n := \{a \in A_n \mid \pi(a) \in \mathring{P}_n \ \&\ \mathrm{mtp}(a) = 0\}$.

Remark 2.24. Note that Clauses (2) and (3) imply that for all $m,n<\omega$, $a \in \mathring{A}_m$ and $q \leq \pi(a)$, if $q \in \mathring{P}_n$ then $\Cap(a)(q) \in \mathring{A}_n$.

The next definition is a weakening of [PRS20, Definition 4.11].

Definition 2.25 (Weak Mixing Property)**.** The forking projection $(\Cap,\pi)$ is said to have the *weak mixing property* iff it admits a type $\mathtt{tp}$ satisfying that for all $n<\omega$, $a \in A$, $\vec{r}$, and $p' \leq^0 \pi(a)$, and for every function $g : W_n(\pi(a)) \to \mathbb{A} \downarrow a$, if there exists an ordinal ι such that all of the following hold:[5]

[5]The ordinal ι would help us keep track of the support when appealing to the weak mixing property in an iteration (see, e.g., Lemma 3.10 and Claim 3.11.6).

(1) $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$ is a good enumeration of $W_n(\pi(a))$;
(2) $\langle \pi(g(r_\xi)) \mid \xi < \chi\rangle$ is diagonalizable with respect to $\vec{r}$, as witnessed by p';[6]
(3) for every $\xi < \chi$:
   - if $\xi < \iota$, then $\mathrm{dom}(\mathtt{tp}(g(r_\xi))) = 0$;
   - if $\xi = \iota$, then $\mathrm{dom}(\mathtt{tp}(g(r_\xi))) \geq 1$;
   - if $\xi > \iota$, then $\mathrm{dom}(\mathtt{tp}(g(r_\xi))) > (\sup_{\eta<\xi} \mathrm{dom}(\mathtt{tp}(g(r_\eta)))) + 1$;
(4) for all $\xi \in (\iota, \chi)$ and $i \in [\mathrm{dom}(\mathtt{tp}(a)), \sup_{\eta<\xi} \mathrm{dom}(\mathtt{tp}(g(r_\eta)))]$,
$$\mathtt{tp}(g(r_\xi))(i) \leq \mathrm{mtp}(a),$$
(5) $\sup_{\xi<\chi} \mathrm{mtp}(g(r_\xi)) < \omega$,

then there exists $b \trianglelefteq^0 a$ with $\pi(b) = p'$ such that, for all $q' \in W_n(p')$,
$$\pitchfork(b)(q') \trianglelefteq^0 g(w(\pi(a), q')).$$

Lemma 2.26. *The forking projection $(\pitchfork, \pi)$ from Example 2.15 has the weak mixing property.*

Proof. We attach a type $\mathtt{tp} : A \to {}^{<\mu}\omega$ as follows. For every $a = (p, \alpha) \in A$, with $\alpha > 0$, let $\mathtt{tp}(a)$ be the constant $(\alpha+1)$-sequence whose sole value is 0. Otherwise, let $\mathtt{tp}(a) := \emptyset$. We shall verify that $\mathtt{tp}$ witnesses that $(\pitchfork, \pi)$ has the weak mixing property. To this end, suppose that we are given $n < \omega$, $a \in A$, $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$, $p' \leq^0 \pi(a)$, a function $g : W_n(\pi(a)) \to \mathbb{A} \downarrow a$ and an ordinal ι satisfying Clauses (1)–(4) of Definition 2.25. For each $\xi < \chi$, write $(q_\xi, \alpha_\xi) := g(r_\xi)$. Note that by Clause (4) of Definition 2.23 and Example 2.15, $\alpha_\xi = 0$ for all $\xi < \iota$.

Set $b := (p', \alpha')$, for $\alpha' := \sup_{\iota\leq\xi<\chi} \alpha_\xi$. Clearly, $b \trianglelefteq^0 a$. Note that, by regularity of μ, $\alpha' < \mu$. Now, since p' witnesses that $\langle q_\xi \mid \xi < \chi\rangle$ is diagonalizable, for every $q' \in W_n(p')$, if we let ξ denote the unique index to satisfy $r_\xi = w(\pi(a), q')$, then $q' \leq^0 q_\xi$. As $\alpha' \geq \alpha_\xi$, it altogether follows that $(q', \alpha') = \pitchfork(b)(q') \trianglelefteq^0 g(w(\pi(a), q')) = (q_\xi, \alpha_\xi)$. □

Lemma 2.27. *Suppose that $(\pitchfork, \pi)$ has the weak mixing property and that $(\mathbb{P}, \ell_{\mathbb{P}})$ has property $\mathcal{D}$. Then $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$, as well.*

Proof. Let $a \in A$ and $n < \omega$. Let $\vec{s} = \langle s_\xi \mid \xi < \chi\rangle$ be a good enumeration of $W_n(a)$. By Lemma 2.11 and Definition 2.23(7), it suffices to show that **I** has a winning strategy in $\Game_{\mathbb{A}}(a, \vec{s}, D)$, where $D := \mathring{A}_{\ell_{\mathbb{A}}(a)+n}$. For each $\xi < \chi$, let $r_\xi := \pi(s_\xi)$. By Fact 2.19, $s_\xi = \pitchfork(a)(r_\xi)$, and $\vec{r} := \langle r_\xi \mid \xi < \chi\rangle$ forms a good enumeration of $W_n(\pi(a))$.

Fix any type $\mathtt{tp}$ witnessing the weak mixing property of $(\pitchfork, \pi)$. We shall describe a winning strategy for **I** in the game $\Game_{\mathbb{A}}(a, \vec{s}, D)$ by producing sequences of the form $\langle (p_\eta, a_\eta, b_\eta, q_\eta) \mid \eta < \xi\rangle$, where $\langle (a_\eta, b_\eta) \mid \eta < \xi\rangle$ is an initial play (consisting of ξ rounds) in the game $\Game_{\mathbb{A}}(a, \vec{s}, D)$, and $\langle (p_\eta, q_\eta) \mid \eta < \xi\rangle$ is an initial play in the game $\Game_{\mathbb{P}}(\pi(a), \vec{r})$. Roughly speaking, the idea

[6] In particular, $\pi(g(r_\xi)) \leq^0 r_\xi$ and $\ell_{\mathbb{A}}(g(r_\xi)) = \ell_{\mathbb{A}}(a) + n$ for every $\xi < \chi$

is to design the moves of **I** (i.e., the a_η's) so that they force **II** to play conditions b_η in such a way that the map $g\colon s_\eta \mapsto b_\eta$ satisfies the requirements of Definition 2.25; most notably, Clauses (3) and (4). To comply with this we shall do suitable stretches when defining the conditions a_η's.

▶ For $\xi = 0$, we first play a condition p_0 according to the winning strategy for **I** in the game $\Game_{\mathbb{P}}(\pi(a), \vec{r})$. In particular, $p_0 \leq^0 \pi(a)$. As p_0 is compatible with r_0, fix a condition $r' \leq p_0, r_0$, and note that it follows from Definition 2.13(2) that $\pitchfork(a)(r') \trianglelefteq \pitchfork(a)(p_0), \pitchfork(a)(r_0)$. Now set $\bar{\alpha}_0 := \mathrm{dom}(\mathsf{tp}(a))+1$ and $a_0 := \pitchfork(a)(p_0)^{\frown\bar{\alpha}_0}$. By Definition 2.23(5), $a_0 \trianglelefteq^0 a$, $\pi(a_0) = p_0$, and it can also be shown that a_0 is compatible with s_0. Indeed, by Clauses (5) and (6) of Definition 2.23, we infer that:

- $a_0 \trianglelefteq^\pi \pitchfork(a)(p_0)$;
- $\mathrm{dom}(\mathsf{tp}(a_\xi)) = \bar{\alpha}_0 + 1$;
- $\mathsf{tp}(a_0)(i) \leq \mathrm{mtp}(\pitchfork(a)(p_0))$, whenever $\mathrm{dom}(\mathsf{tp}(\pitchfork(a)(p_0))) \leq i \leq \bar{\alpha}_0$;
- $\pitchfork(a)(r')^{\frown\bar{\alpha}_0} \trianglelefteq \pitchfork(a)(p_0)^{\frown\bar{\alpha}_0} = a_0$ and $\pitchfork(a)(r')^{\frown\bar{\alpha}_0} \trianglelefteq \pitchfork(a)(r') \trianglelefteq s_0$.

Thus, a_0 is compatible with s_0. Next, let **II** play $b_0 \in D$ at will, subject to ensuring that $b_0 \trianglelefteq a_0$ and $b_0 \trianglelefteq^0 s_0$. Finally, let $q_0 := \pi(b_0)$.

▶ Suppose that $\xi < \chi$ is nonzero and that $\langle (p_\eta, a_\eta, b_\eta, q_\eta) \mid \eta < \xi\rangle$ has already been defined. Let p_ξ be given by the winning strategy for **I** in the game $\Game_{\mathbb{P}}(\pi(a), \vec{r})$ with respect to the initial play $\langle (p_\eta, q_\eta) \mid \eta < \xi\rangle$. As in the previous case, we may fix a condition r' such that $\pitchfork(a)(r') \trianglelefteq \pitchfork(a)(p_\xi), s_\xi$.

Set $\bar{\alpha}_\xi := (\sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}(b_\eta))) + 1$. Then, by Clauses (5) and (6) of Definition 2.23, we may let $a_\xi := \pitchfork(a)(p_\xi)^{\frown\bar{\alpha}_\xi}$, and argue as before that a_ξ is compatible with s_ξ. Also, note that $a_\xi \trianglelefteq^0 a$ and $\pi(a_\xi) = p_\xi$. Next, let **II** play any $b_\xi \in D$ such that $b_\xi \trianglelefteq a_\xi$ and $a_\xi \trianglelefteq^0 s_\xi$. Finally, let $q_\xi := \pi(b_\xi)$.

At the end of the game, we have produced a sequence $\langle (p_\xi, a_\xi, b_\xi, q_\xi) \mid \xi < \chi\rangle$. Since $\langle (p_\xi, q_\xi) \mid \xi < \chi\rangle$ is the outcome of a $\Game_{\mathbb{P}}(\pi(a), \vec{r})$-game in which **I** played according to a winning strategy, we may fix $p' \leq^0 \pi(a)$ witnessing that $\langle q_\xi \mid \xi < \chi\rangle$ is diagonalizable.

It follows that if we define a function $g : W_n(\pi(a)) \to D$ via $g(r_\xi) := b_\xi$, then all the requirements of Definition 2.25 are fulfilled with respect to $\iota := 0$ (Note that we have secured that $\mathrm{dom}(\mathsf{tp}(a_\xi)) > 0$ for all $\xi < \chi$). For instance, to see that Clause (4) of Definition 2.25 holds, notice that by Clauses (2) and (5) of Definition 2.23, for all $\xi < \chi$ and $i \in [\mathrm{dom}(\mathsf{tp}(a)), \mathrm{dom}(\mathsf{tp}(a_\xi)))$,

$$\mathsf{tp}(b_\xi)(i) \leq \mathsf{tp}(a_\xi)(i) \leq \mathrm{mtp}(\pitchfork(a)(p_\xi)) \leq \mathrm{mtp}(a).$$

In effect, we may pick $b \trianglelefteq^0 a$ with $\pi(b) = p'$ such that for all $q' \in W_n(p')$,

$$\pitchfork(b)(q') \trianglelefteq^0 g(w(\pi(a), q')).$$

By definition, for each $q' \in W_n(p')$, $g(w(\pi(a), q')) = b_\xi$, where ξ is the unique index such that $\pi(s_\xi) = w(\pi(a), q')$. Therefore, invoking Proposition 2.22 we infer that b diagonalizes $\langle b_\xi \mid \xi < \chi\rangle$, as desired. □

Corollary 2.28. *If $(\mathbb{P}, \ell_{\mathbb{P}})$ has property $\mathcal{D}$, and $(\pitchfork, \pi)$ has the weak mixing property, then $(\mathbb{A}, \ell_{\mathbb{A}})$ has the* CPP.

Proof. By Lemmas 2.21 and 2.27. □

Lemma 2.29. *Suppose that $(\pitchfork, \pi)$ is as in Setup 2 or, just a pair of maps satisfying Clauses (1), (2), (5) and (7) of Definition 2.13.*

Let $n < \omega$. If $(\pitchfork, \pi)$ admits a type, and $\mathring{\mathbb{A}}_n$ is defined according to the last clause of Definition 2.23, if $\mathring{\mathbb{A}}^{\pi}_n$ is κ_n-directed-closed, then so is $\mathring{\mathbb{A}}_n$.

Proof. The proof is very similar to that of [PRS20, Lemma 4.6], bearing Remark 2.24 in mind. □

3. Iteration Scheme

In this section, we present our iteration scheme for Σ-Prikry posets. Throughout the section, assume that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals. Denote $\kappa := \sup_{n<\omega} \kappa_n$. Also, assume that μ is some cardinal satisfying $\mu^{<\mu} = \mu$, so that $|H_\mu| = \mu$.

The following convention will be applied hereafter:

Convention 3.1. For all ordinals $\alpha \leq \delta \leq \mu^+$:

(1) $\emptyset_\delta := \delta \times \{\emptyset\}$ denotes the δ-sequence with constant value $\emptyset$;
(2) For an α-sequence p and a δ-sequence q, $p * q$ denotes the unique δ-sequence satisfying that for all $\beta < \delta$:
$$(p * q)(\beta) = \begin{cases} q(\beta), & \text{if } \alpha \leq \beta < \delta; \\ p(\beta), & \text{otherwise.} \end{cases}$$
(3) Let $\mathbb{P}_\delta := (P_\delta, \leq_\delta)$ and $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)$ be forcing posets such that $P_\delta \subseteq {}^{\delta}H_{\mu^+}$ and $P_\alpha \subseteq {}^{\alpha}H_{\mu^+}$. Also, assume $p \mapsto p \restriction \alpha$ defines a projection between $\mathbb{P}_\delta$ and $\mathbb{P}_\alpha$. We denote by $i^\delta_\alpha : V^{\mathbb{P}_\alpha} \to V^{\mathbb{P}_\delta}$ the map defined by recursion over the rank of each $\mathbb{P}_\alpha$-name σ as follows:
$$i^\delta_\alpha(\sigma) := \{(i^\delta_\alpha(\tau), p * \emptyset_\delta) \mid (\tau, p) \in \sigma\}.$$

Our iteration scheme requires three building blocks:

Building Block I. We are given a Σ-Prikry triple $(\mathbb{Q}, \ell, c)$ such that $\mathbb{Q} = (Q, \leq_Q)$ is a subset of H_{μ^+}, $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q} \check{\mu} = \kappa^+$ and $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q}$ "κ is singular".[7] Additionally, we assume that $(\mathbb{Q}, \ell)$ has property $\mathcal{D}$. To streamline the matter, we also require that $\mathbb{1}_\mathbb{Q}$ be equal to $\emptyset$.

Building Block II. For every Σ-Prikry triple $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ having property $\mathcal{D}$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+}, $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} \check{\mu} = \kappa^+$ and $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ "κ is singular", every $r^\star \in P$, and every $\mathbb{P}$-name $z \in H_{\mu^+}$, we are given a corresponding Σ-Prikry triple $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ such that:

(a) $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ admits a forking projection $(\pitchfork, \pi)$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ that has the weak mixing property;[8]

[7] At the behest of the referee, we stress that the last hypothesis plays a rather isolated role; see Footnote 20 on page 34.

[8] So, by Lemma 2.27, $(\mathbb{A}, \ell_\mathbb{A})$ has property $\mathcal{D}$, as well.

(b) for each $n<\omega$, $\mathring{\mathbb{A}}^{\pi}_n$ is κ_n-directed-closed;[9]
(c) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu}=\kappa^+$;
(d) $\mathbb{A}=(A,\trianglelefteq)$ is a subset of H_{μ^+}.

By Lemma 2.18, we may streamline the matter, and also require that:

(e) each element of A is a pair (x,y) with $\pi(x,y)=x$;
(f) for every $a\in A$, $\lceil\pi(a)\rceil^{\mathbb{A}}=(\pi(a),\emptyset)$;
(g) for every $p,q\in P$, if $c_{\mathbb{P}}(p)=c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p\rceil^{\mathbb{A}})=c_{\mathbb{A}}(\lceil q\rceil^{\mathbb{A}})$.

Building Block III. We are given a function $\psi:\mu^+\to H_{\mu^+}$.

Goal 3.2. Our goal is to define a system $\langle(\mathbb{P}_\delta,\ell_\delta,c_\delta,\langle\pitchfork_{\delta,\gamma}|\ \gamma\leq\delta\rangle)\mid\delta\leq\mu^+\rangle$ in such a way that for all $\gamma\leq\delta\leq\mu^+$:

(i) $\mathbb{P}_\delta$ is a poset $(P_\delta,\leq_\delta)$, $P_\delta\subseteq{}^{\delta}H_{\mu^+}$, and, for all $p\in P_\delta$, $|B_p|<\mu$, where $B_p:=\{\beta+1\mid\beta\in\mathrm{dom}(p)\ \&\ p(\beta)\neq\emptyset\}$;
(ii) The map $\pi_{\delta,\gamma}:P_\delta\to P_\gamma$ defined by $\pi_{\delta,\gamma}(p):=p\restriction\gamma$ forms a projection from $\mathbb{P}_\delta$ to $\mathbb{P}_\gamma$ and $\ell_\delta=\ell_\gamma\circ\pi_{\delta,\gamma}$;
(iii) $\mathbb{P}_0$ is a trivial forcing, $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I, and $\mathbb{P}_{\delta+1}$ is isomorphic to $\mathbb{A}$ given by Building Block II when invoked with $(\mathbb{P}_\delta,\ell_\delta,c_\delta)$ and a pair $(r^\star,z)$ which is decoded from $\psi(\delta)$;
(iv) If $\delta>0$, then $(\mathbb{P}_\delta,\ell_\delta,c_\delta)$ is a Σ-Prikry triple having property $\mathcal{D}$ whose greatest element is $\emptyset_\delta$, $\ell_\delta=\ell_1\circ\pi_{\delta,1}$, and $\emptyset_\delta\Vdash_{\mathbb{P}_\delta}\check{\mu}=\kappa^+$;
(v) If $0<\gamma<\delta\leq\mu^+$, then $(\pitchfork_{\delta,\gamma},\pi_{\delta,\gamma})$ is a forking projection from $(\mathbb{P}_\delta,\ell_\delta)$ to $(\mathbb{P}_\gamma,\ell_\gamma)$; in case $\delta<\mu^+$, $(\pitchfork_{\delta,\gamma},\pi_{\delta,\gamma})$ is furthermore a forking projection from $(\mathbb{P}_\delta,\ell_\delta,c_\delta)$ to $(\mathbb{P}_\gamma,\ell_\gamma,c_\gamma)$, and in case $\delta=\gamma+1$, $(\pitchfork_{\delta,\gamma},\pi_{\delta,\gamma})$ has the weak mixing property;
(vi) If $0<\alpha\leq\beta\leq\delta$, then, for all $p\in P_\delta$ and $r\leq_\alpha p\restriction\alpha$, $\pitchfork_{\beta,\alpha}(p\restriction\beta)(r)=(\pitchfork_{\delta,\alpha}(p)(r))\restriction\beta$.

Remark 3.3. Note the asymmetry between the cases $\delta<\mu^+$ and $\delta=\mu^+$:

(1) By Clause (i), we will have that $\mathbb{P}_\delta\subseteq H_{\mu^+}$ for all $\delta<\mu^+$, but $\mathbb{P}_{\mu^+}\not\subseteq H_{\mu^+}$. Still, $\mathbb{P}_{\mu^+}$ will nevertheless be isomorphic to a subset of H_{μ^+}, as we may identify P_{μ^+} with $\{p\restriction(\sup(B_p)+1)\mid p\in P_{\mu^+}\}$.
(2) Clause (v) puts a weaker assertion for $\delta=\mu^+$. In order to avoid trivialities, let us assume that μ^+-many stages in our iteration $\mathbb{P}_{\mu^+}$ are non-trivial. To see the restriction in Clause (v) is necessary note that, by the pigeonhole principle, there must exist two conditions $p,q\in P_{\mu^+}$ and an ordinal $\gamma<\mu^+$ for which $c_{\mu^+}(p)=c_{\mu^+}(q)$, $B_p\subseteq\gamma$, but $B_q\not\subseteq\gamma$. Now, towards a contradiction, assume there is a map $\pitchfork$ such that $(\pitchfork,\pi_{\mu^+,\gamma})$ forms a forking projection from $(\mathbb{P}_{\mu^+},\ell_{\mu^+},c_{\mu^+})$ to $(\mathbb{P}_\gamma,\ell_\gamma,c_\gamma)$. By Definition 2.13(8), then, $c_\gamma(p\restriction\gamma)=c_\gamma(q\restriction\gamma)$, so that by Definition 2.3(3), we should be able to pick $r\in(P_\gamma)^{p\restriction\gamma}_0\cap(P_\gamma)^{q\restriction\gamma}_0$, and then by Definition 2.13(8), $\pitchfork(p)(r)=\pitchfork(q)(r)$. Finally,

[9] $\mathring{A}_n$ denotes the poset of Definition 2.23(7) regarded with respect to the type witnessing Clause (a) of Building Block II.

as $B_p \subseteq \gamma$, $p = \lceil p \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$,[10] so that, by Definition 2.13(6), $\pitchfork(p)(r) = \lceil r \rceil^{\mathbb{P}_{\mu^+}}$. But then $\pitchfork(q)(r) = \lceil r \rceil^{\mathbb{P}_{\mu^+}}$, so that, by Definition 2.13(6), $q = \lceil q \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$, contradicting the fact that $B_q \not\subseteq \gamma$.

3.1. **Defining the iteration.** For every $\delta < \mu^+$, fix an injection $\phi_\delta : \delta \to \mu$. As $|H_\mu| = \mu$, by the Engelking-Karłowicz theorem, we may also fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from μ^+ to H_μ such that for every function $e : C \to H_\mu$ with $C \in [\mu^+]^{<\mu}$, there is $i < \mu$ such that $e \subseteq e^i$.

The upcoming definition is by recursion on $\delta \leq \mu^+$, and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

▶ Let $\mathbb{P}_0 := (\{\emptyset\}, \leq_0)$ be the trivial forcing. Let ℓ_0 and c_0 be the constant function $\{(\emptyset, \emptyset)\}$, and let $\pitchfork_{0,0}$ be the constant function $\{(\emptyset, \{(\emptyset, \emptyset)\})\}$, so that $\pitchfork_{0,0}(\emptyset)$ is the identity map.

▶ Let $\mathbb{P}_1 := (P_1, \leq_1)$, where $P_1 := {}^1Q$ and $p \leq_1 p'$ iff $p(0) \leq_Q p'(0)$. Define ℓ_1 and c_1 by stipulating $\ell_1(p) := \ell(p(0))$ and $c_1(p) = c(p(0))$. For all $p \in P_1$, let $\pitchfork_{1,0}(p) : \{\emptyset\} \to \{p\}$ be the constant function, and let $\pitchfork_{1,1}(p)$ be the identity map.

▶ Suppose $\delta < \mu^+$ and that $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle \pitchfork_{\beta,\gamma} \mid \gamma \leq \beta \rangle) \mid \beta \leq \delta \rangle$ has already been defined. We now define the triple $(\mathbb{P}_{\delta+1}, \ell_{\delta+1}, c_{\delta+1})$ and the sequence of maps $\langle \pitchfork_{\delta+1,\gamma} \mid \gamma \leq \delta + 1 \rangle$.

▶▶ If $\psi(\delta)$ happens to be a triple (β, r, σ), where $\beta < \delta$, $r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, then we appeal to Building Block II with $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$, $r^\star := r * \emptyset_\delta$ and $z := i^\delta_\beta(\sigma)$ to get a corresponding Σ-Prikry poset $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$.

▶▶ Otherwise, we obtain $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ by appealing to Building Block II with $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$, $r^\star := \emptyset_\delta$ and $z := \emptyset$.

In both cases, we also obtain a forking projection $(\pitchfork, \pi)$ from $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ to $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$. Furthermore, each condition in $\mathbb{A} = (A, \trianglelefteq)$ is a pair (x, y) with $\pi(x, y) = x$, and, for every $p \in P_\delta$, $\lceil p \rceil^\mathbb{A} = (p, \emptyset)$. Now, define $\mathbb{P}_{\delta+1} := (P_{\delta+1}, \leq_{\delta+1})$ by letting $P_{\delta+1} := \{x^\frown\langle y \rangle \mid (x, y) \in A\}$, and then let $p \leq_{\delta+1} p'$ iff $(p \restriction \delta, p(\delta)) \trianglelefteq (p' \restriction \delta, p'(\delta))$. Put $\ell_{\delta+1} := \ell_1 \circ \pi_{\delta+1,1}$ and define $c_{\delta+1} : P_{\delta+1} \to H_\mu$ via $c_{\delta+1}(p) := c_\mathbb{A}(p \restriction \delta, p(\delta))$.

Next, let $p \in P_{\delta+1}$, $\gamma \leq \delta + 1$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\pitchfork_{\delta+1,\gamma}(p)(r)$. For $\gamma = \delta + 1$, let $\pitchfork_{\delta+1,\gamma}(p)(r) := r$, and for $\gamma \leq \delta$, let

$$(*) \qquad \pitchfork_{\delta+1,\gamma}(p)(r) := x^\frown\langle y \rangle \text{ iff } \pitchfork(p \restriction \delta, p(\delta))(\pitchfork_{\delta,\gamma}(p \restriction \delta)(r)) = (x, y).$$

▶ Suppose $\delta \in \mathrm{acc}(\mu^+ + 1)$, and that $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle \pitchfork_{\beta,\gamma} \mid \gamma \leq \beta \rangle) \mid \beta < \delta \rangle$ has already been defined. Define $\mathbb{P}_\delta := (P_\delta, \leq_\delta)$ by letting P_δ be all δ-sequences p such that $|B_p| < \mu$ and $\forall \beta < \delta (p \restriction \beta \in P_\beta)$. Let $p \leq_\delta q$ iff $\forall \beta < \delta (p \restriction \beta \leq_\beta q \restriction \beta)$. Let $\ell_\delta := \ell_1 \circ \pi_{\delta,1}$. Next, we define $c_\delta : P_\delta \to H_\mu$, as follows.

[10] This is a consequence of the fact that $p = (p \restriction \gamma) * \emptyset_{\mu^+} = \lceil p \restriction \gamma \rceil^{\mathbb{P}_{\mu^+}}$. See the discussion at the beginning of Lemma 3.6.

►► If $\delta < \mu^+$, then, for every $p \in P_\delta$, let

$$c_\delta(p) := \{(\phi_\delta(\gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\}.$$

►► If $\delta = \mu^+$, then, given $p \in P_\delta$, first let $C := \mathrm{cl}(B_p)$, then define a function $e : C \to H_\mu$ by stipulating:

$$e(\gamma) := (\phi_\gamma[C \cap \gamma], c_\gamma(p \restriction \gamma)),$$

and then let $c_\delta(p) := i$ for the least $i < \mu$ such that $e \subseteq e^i$.

Finally, let $p \in P_\delta$, $\gamma \leq \delta$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\pitchfork_{\delta,\gamma}(p)(r)$. For $\gamma = \delta$, let $\pitchfork_{\delta,\gamma}(p)(r) := r$, and for $\gamma < \delta$, let $\pitchfork_{\delta,\gamma}(p)(r) := \bigcup\{\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \delta\}$.

Convention 3.4. Even though $(\mathbb{P}_0, \ell_0)$ is not a graded poset, in order to smooth up inductive claims that come later, we define $\leq_0^0$ to be $\leq_0$, and likewise, for every $p \in P_0$, we interpret $(P_0)_0^p$ as $\{q \in P_0 \mid q \leq_0^0 p\}$.

3.2. **Verification.** We now verify that for all $\delta \leq \mu^+$, $(\mathbb{P}_\delta, \ell_\delta, c_\delta, \langle \pitchfork_{\delta,\gamma} | \ \gamma \leq \delta\rangle)$ fulfills requirements (i)–(vi) of Goal 3.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. We commence with an expanded version of Clause (vi).

Lemma 3.5. *For all $\alpha \leq \delta \leq \mu^+$, $p \in P_\delta$ and $r \in P_\alpha$ with $r \leq_\alpha p \restriction \alpha$, if we let $q := \pitchfork_{\delta,\alpha}(p)(r)$, then:*

(1) $q \restriction \beta = \pitchfork_{\beta,\alpha}(p \restriction \beta)(r)$ for all $\beta \in [\alpha, \delta]$;
(2) $B_q = B_p \cup B_r$;
(3) $q \restriction \alpha = r$;
(4) If $\alpha = 0$, then $q = p$;
(5) $p = (p \restriction \alpha) * \emptyset_\delta$ iff $q = r * \emptyset_\delta$;
(6) for all $p' \leq_\delta^0 p$, if $r \leq_\alpha^0 p' \restriction \alpha$, then $\pitchfork_{\delta,\alpha}(p')(r) \leq_\delta \pitchfork_{\delta,\alpha}(p)(r)$.

Proof. Clause (3) follows from Clause (1) and the fact that $\pitchfork_{\alpha,\alpha}(p \restriction \alpha)$ is the identity function. Clause (5) follows from Clauses (2) and (3).

We now prove Clauses (1), (2), (4) and (6) by induction on $\delta \leq \mu^+$:

- The case $\delta = 0$ is trivial, since, in this case, all the conditions under consideration (and their corresponding B-sets) are empty, and all the maps under consideration are the identity.
- The case $\delta = 1$ follows from the fact that, by definition, $\pitchfork_{1,0}(p)(r) = p$ and $\pitchfork_{1,1}(p)(r) = r$.
- Suppose $\delta \geq 2$ is a successor ordinal, say $\delta = \delta' + 1$, and that the claim holds for δ'. Fix arbitrary $\alpha \leq \delta$, $p \in P_\delta$ and $r \in P_\alpha$ with $r \leq_\alpha p \restriction \alpha$. Denote $q := \pitchfork_{\delta,\alpha}(p)(r)$. Recall that $\mathbb{P}_\delta = \mathbb{P}_{\delta'+1}$ was defined by feeding $(\mathbb{P}_{\delta'}, \ell_{\delta'}, c_{\delta'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ along with a forking projection $(\pitchfork, \pi)$, such that each condition in the poset $\mathbb{A} = (A, \trianglelefteq)$ is a pair (x, y) with $\pi(x, y) = x$. Furthermore, by the definition of $\pitchfork_{\delta,\alpha}$, $q = \pitchfork_{\delta,\alpha}(p)(r)$ is equal to $x^\frown\langle y\rangle$, where

$$(x, y) := \pitchfork(p \restriction \delta', p(\delta'))(\pitchfork_{\delta',\alpha}(p \restriction \delta')(r)).$$

In particular, $q \restriction \delta' = x = \pi(\Cap(p \restriction \delta', p(\delta'))(\Cap_{\delta',\alpha}(p \restriction \delta')(r)))$, which, by Definition 2.13(5), is equal to $\Cap_{\delta',\alpha}(p \restriction \delta')(r)$.

(1) It follows that, for all $\beta \in [\alpha, \delta)$,

$$q \restriction \beta = (q \restriction \delta') \restriction \beta = \Cap_{\delta',\alpha}(p \restriction \delta')(r) \restriction \beta = \Cap_{\beta,\alpha}(p \restriction \beta)(r),$$

where the rightmost equality follows from the induction hypothesis. In addition, the case $\beta = \delta$ is trivial.

(2) To avoid trivialities, assume $\alpha < \delta$. By Clause (1), $q \restriction \delta' = \Cap_{\delta,\alpha}(p \restriction \delta')(r)$. So, by the induction hypothesis, $B_{q\restriction\delta'} = B_{p\restriction\delta'} \cup B_r$, and we are left with showing that $\delta \in B_q$ iff $\delta \in B_p$. As $q \leq_\delta p$, we have $B_q \supseteq B_p$, so the forward implication is clear. Finally, if $\delta \notin B_p$, then $p(\delta') = \emptyset$, and hence

$$(x, y) = \Cap(p \restriction \delta', \emptyset)(\Cap_{\delta',\alpha}(p \restriction \delta')(r)).$$

It thus follows from Clause (f) of Building Block II together with the fact that $\Cap$ satisfies Clause (6) of Definition 2.13 that $(x, y) = (\Cap_{\delta',\alpha}(p \restriction \delta')(r), \emptyset)$. Recalling that $q = x^\frown\langle y\rangle$, we conclude that $\delta \notin B_q$, as desired.

(4) If $\alpha = 0$, then, by the induction hypothesis, $\Cap_{\delta',0}(p \restriction \delta')(r) = p \restriction \delta'$, so that

$$\begin{aligned}(x, y) = &\ \Cap(p \restriction \delta', p(\delta'))(\Cap_{\delta',0}(p \restriction \delta')(r))\\ = &\ \Cap(p \restriction \delta', p(\delta'))(p \restriction \delta')\\ = &\ (p \restriction \delta', p(\delta')) = (x, y),\end{aligned}$$

where the rightmost equality follows from Lemma 2.17. Altogether, $q = x^\frown\langle y\rangle = p$.

(6) To avoid trivialities, assume that $\Cap_{\delta,\alpha}(p')(r) \neq \Cap_{\delta,\alpha}(p)(r)$, so that $\alpha < \delta$. By Clause (4), we may also assume that $0 < \alpha$. Fix $p' \leq^0_\delta p$ with $r \leq^0_\alpha p' \restriction \alpha$. By the definition of $\leq_{\delta'+1}$, proving $\Cap_{\delta,\alpha}(p')(r) \leq_\delta \Cap_{\delta,\alpha}(p)(r)$ amounts to verifying that $(x', y') \trianglelefteq (x, y)$, where

$$(x', y') := \Cap(p' \restriction \delta', p'(\delta'))(\Cap_{\delta',\alpha}(p' \restriction \delta')(r)).$$

Now, by the induction hypothesis, $\Cap_{\delta',\alpha}(p' \restriction \delta')(r) \leq_{\delta'} \Cap_{\delta',\alpha}(p \restriction \delta')(r)$. So, since $\Cap(p \restriction \delta', p(\delta'))$ is order-preserving, it suffices to prove that

$$(x', y') \trianglelefteq \Cap(p \restriction \delta', p(\delta'))(\Cap_{\delta',\alpha}(p' \restriction \delta')(r)).$$

Denote $a := (p\restriction\delta', p(\delta'))$ and $a' := (p'\restriction\delta', p'(\delta'))$. Then, by Clause (7) of Definition 2.13, indeed

$$\Cap(a')(\Cap_{\delta',\alpha}(p' \restriction \delta')(r)) \trianglelefteq \Cap(a)(\Cap_{\delta',\alpha}(p' \restriction \delta')(r)).$$

► Suppose $\delta \in \mathrm{acc}(\mu^+ + 1)$ is an ordinal such that, for all $\delta' < \delta$, $\beta \in [\alpha, \delta']$, $p \in P_{\delta'}$ and $r \in P_\alpha$ with $r \leq_\alpha p \restriction \alpha$,

$$\Cap_{\beta,\alpha}(p \restriction \beta)(r) = (\Cap_{\delta',\alpha}(p \restriction \delta')(r)) \restriction \beta.$$

Fix arbitrary $\alpha \leq \delta$, $p \in P_\delta$ and $r \in P_\alpha$ with $r \leq_\alpha p \restriction \alpha$. Denote $q := \pitchfork_{\delta,\alpha}(p)(r)$. By our definition of $\pitchfork_{\delta,\alpha}$ at the limit stage, we have:

$$q = \bigcup\{\pitchfork_{\beta,\alpha}(p \restriction \beta)(r) \mid \alpha \leq \beta < \delta\}.$$

By the induction hypothesis, $\langle \pitchfork_{\beta,\alpha}(p \restriction \beta)(r) \mid \alpha \leq \beta < \delta\rangle$ is a $\subseteq$-increasing sequence, and $B_{\pitchfork_{\beta,\alpha}(p\restriction\beta)(r)} = B_{p\restriction\beta} \cup B_r$ whenever $\alpha \leq \beta < \delta$. It thus follows that q is a legitimate condition, and Clauses (1), (2), (4) and (6) are satisfied. □

Our next task is to verify Clauses (ii) and (v) of Goal 3.2:

Lemma 3.6. *Suppose that $\delta \leq \mu^+$ is such that for all nonzero $\gamma < \delta$, $(\mathbb{P}_\gamma, c_\gamma, \ell_\gamma)$ is Σ-Prikry. Then:*

- *for all nonzero $\gamma \leq \delta$, $(\pitchfork_{\delta,\gamma}, \pi_{\delta,\gamma})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_\gamma, \ell_\gamma)$, where $\pi_{\delta,\gamma}$ is defined as in Goal 3.2(ii);*
- *if $\delta < \mu^+$, then $(\pitchfork_{\delta,\gamma}, \pi_{\delta,\gamma})$ is furthermore a forking projection from $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$*
- *if $\delta = \gamma + 1 > 1$, then $(\pitchfork_{\delta,\gamma}, \pi_{\delta,\gamma})$ has the weak mixing property.*

Proof. Let us go over the clauses of Definition 2.13.

Clause (5) is covered by Lemma 3.5(3), and Clause (7) is covered by Lemma 3.5(6). Clause (3) is obvious, since for all nonzero $\gamma < \delta$ and $p \in P_\gamma$, a straight-forward verification makes it clear that $p{*}\emptyset_\delta$ is the greatest element of $\{q \in P_\delta \mid \pi_{\delta,\gamma}(q) = p\}$. In effect, Clause (6) follows from Lemma 3.5(5).

Thus, we are left with verifying Clauses (1),(2),(4) and (8). The next claim takes care of the first three.

Claim 3.6.1. *For all nonzero $\gamma \leq \delta$ and $p \in P_\delta$:*

(1) $\pi_{\delta,\gamma}$ forms a projection from $\mathbb{P}_\delta$ to $\mathbb{P}_\gamma$, and $\ell_\delta = \ell_\gamma \circ \pi_{\delta,\gamma}$;
(2) $\pitchfork_{\delta,\gamma}(p)$ is an order-preserving function from $(\mathbb{P}_\gamma \downarrow (p \restriction \gamma), \leq_\gamma)$ to $(\mathbb{P}_\delta \downarrow p, \leq_\delta)$;
(3) for all $n, m < \omega$ and $q \leq_\delta^{n+m} p$, $m(p,q)$ exists and, furthermore,

$$m(p,q) = \pitchfork_{\delta,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma)).$$

Proof. We commence by proving (2) and (3) by induction on $\delta \leq \mu^+$:

- The case $\delta = 1$ is trivial, since, in this case, $\gamma = \delta$.
- Suppose $\delta = \delta' + 1$ is a successor ordinal and that the claim holds for δ'. Let $\gamma \leq \delta$ and $p \in P_\delta$ be arbitrary. To avoid trivialities, assume $\gamma < \delta$. By the induction hypothesis, $\pitchfork_{\delta',\gamma}(p \restriction \delta')$ is an order-preserving function from $\mathbb{P}_\gamma \downarrow (p \restriction \gamma)$ to $\mathbb{P}_{\delta'} \downarrow (p \restriction \delta')$.

  Recall that $\mathbb{P}_\delta = \mathbb{P}_{\delta'+1}$ was defined by feeding $(\mathbb{P}_{\delta'}, \ell_{\delta'}, c_{\delta'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along with the pair $(\pitchfork, \pi)$. Now, as $\pitchfork(p \restriction \delta', p(\delta'))$ and $\pitchfork_{\delta',\gamma}(p \restriction \delta')$ are both order-preserving, the very definition of $\pitchfork_{\delta,\gamma}(p \restriction \gamma)$ and $\leq_{\delta'+1}$ implies that $\pitchfork_{\delta,\gamma}(p \restriction \gamma)$ is order-preserving. In addition, as (x, y) is a condition in $\mathbb{A}$ iff $x^\frown\langle y\rangle \in P_\delta$ and as $\pitchfork(p \restriction \delta', p(\delta'))$ is an order-preserving function from $\mathbb{P}_{\delta'} \downarrow (p \restriction \delta')$ to $\mathbb{A} \downarrow (p \restriction \delta', p(\delta'))$, we infer that, for all $r \leq_\gamma p \restriction \gamma$, $\pitchfork_{\delta,\gamma}(p \restriction \gamma)(r)$ is in $\mathbb{P}_\delta \downarrow p$.

Let $q \leq^{n+m}_{\delta} p$ for some $n, m < \omega$. Let

$$(x, y) := m((p \restriction \delta', p(\delta')), (q \restriction \delta', q(\delta'))).$$

Trivially, $m(p, q)$ exists and is equal to $x^\frown\langle y\rangle$. We need to show that $m(p, q) = \pitchfork_{\delta,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$. By Definition 2.13(4),

$$(x, y) = \pitchfork(p \restriction \delta', p(\delta'))(m(p \restriction \delta', q \restriction \delta')).$$

By the induction hypothesis,

$$m(p \restriction \delta', q \restriction \delta') = \pitchfork_{\delta',\gamma}(p \restriction \delta')(m(p \restriction \gamma, q \restriction \gamma)),$$

and so it follows that

$$(x, y) = \pitchfork(p \restriction \delta', p(\delta'))(\pitchfork_{\delta',\gamma}(p \restriction \delta')(m(p \restriction \gamma, q \restriction \gamma))).$$

Thus, by the definition of $\pitchfork_{\delta,\gamma}$ and the above equation, we have that $\pitchfork_{\delta,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$ is indeed equal to $x^\frown\langle y\rangle$.

► Suppose $\delta \in \mathrm{acc}(\mu^+ + 1)$ is an ordinal for which the claim holds below δ. Let $\gamma \leq \delta$ and $p \in P_\delta$ be arbitrary. To avoid trivialities, assume $\gamma < \delta$. By Lemma 3.5(1), for every $r \in \mathbb{P}_\gamma \downarrow (p \restriction \gamma)$:

$$\pitchfork_{\delta,\gamma}(p)(r) = \bigcup_{\gamma \leq \delta' < \delta} \pitchfork_{\delta',\gamma}(p \restriction \delta')(r).$$

As for all $q, q' \in P_\delta$, $q \leq_\delta q'$ iff $\forall \delta' < \delta(q \restriction \delta' \leq_{\delta'} q' \restriction \delta')$, the induction hypothesis implies that $\pitchfork_{\delta,\gamma}(p)$ is an order-preserving function from $\mathbb{P}_\gamma \downarrow (p \restriction \gamma)$ to $\mathbb{P}_\delta \downarrow p$;

Finally, let $q \leq_\delta p$; we shall show that $m(p, q)$ exists and is, in fact, equal to $\pitchfork_{\delta,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma))$. By Lemma 3.5(1) and the induction hypothesis,

$$\pitchfork_{\delta,\gamma}(p)(m(p \restriction \gamma, q \restriction \gamma)) = \bigcup_{\gamma \leq \delta' < \delta} m(p \restriction \delta', q \restriction \delta'),$$

call it r. We shall show that r plays the role of $m(p, q)$.

By the definition of $\leq_\delta$, it is clear that $q \leq^m_\delta r \leq^n_\delta p$, so it remains to show that it is the greatest condition in $(P^p_\delta)_n$ to satisfy this. Fix an arbitrary $s \in (P^p_\delta)_n$ with $q \leq^m_\delta s$. For each $\delta' < \delta$, $q \restriction \delta' \leq^m_{\delta'} s \restriction \delta' \leq^n_{\delta'} p \restriction \delta'$, so that $s \restriction \delta' \leq_{\delta'} m(p \restriction \delta', q \restriction \delta')$, and thus $s \leq_\delta r$. Altogether this shows that $r = m(p, q)$.

This completes the proof of Clauses (2) and (3) above.

We are left to prove (1). The case $\gamma = \delta$ is trivial, so assume $\gamma < \delta$. Clearly, $\pi_{\delta,\gamma}$ is order-preserving and also $\pi_{\delta,\gamma}(\emptyset_\delta) = \emptyset_\gamma$. Let $p \in P_\delta$ and $q \in P_\gamma$ be such that $q \leq_\gamma \pi_{\delta,\gamma}(p)$. Set $q^* := \pitchfork_{\delta,\gamma}(p)(q)$. By Lemma 3.5(3), $\pi_{\delta,\gamma}(q^*) = q$ and by Clause (2) of this claim, $q^* \leq_\delta p$. Altogether, $\pi_{\delta,\gamma}$ is indeed a projection. For the second part, recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\beta,1}$, hence $\ell_\delta = \ell_1 \circ \pi_{\delta,1} = \ell_1 \circ (\pi_{\gamma,1} \circ \pi_{\delta,\gamma}) = (\ell_1 \circ \pi_{\gamma,1}) \circ \pi_{\delta,\gamma} = \ell_\gamma \circ \pi_{\delta,\gamma}$. □

We are left with verifying Clause (8) of Definition 2.13 to show that $(\pitchfork_{\delta,\gamma}, \pi_{\delta,\gamma})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$.

Claim 3.6.2. *Suppose $\delta \neq \mu^+$. For all $p, p' \in P_\delta$ with $c_\delta(p) = c_\delta(p')$ and all nonzero $\gamma \leq \delta$:*

- *$c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$, and*
- *$\pitchfork_{\delta,\gamma}(p)(r) = \pitchfork_{\delta,\gamma}(p')(r)$ for every $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{p'\restriction\gamma}$.*

Proof. By induction on $\delta < \mu^+$:

- ▶ The case $\delta = 1$ is trivial, since, in this case, $\gamma = \delta$.
- ▶ Suppose $\delta = \delta' + 1$ is a successor ordinal and that the claim holds for δ'. Fix an arbitrary pair $p, p' \in P_\delta$ with $c_\delta(p) = c_\delta(p')$.

  Recall that $\mathbb{P}_\delta = \mathbb{P}_{\delta'+1}$ was defined by feeding $(\mathbb{P}_{\delta'}, \ell_{\delta'}, c_{\delta'})$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along the pair $(\pitchfork, \pi)$. By the definition of $c_{\delta'+1}$, we have

  $$c_{\mathbb{A}}(p \restriction \delta', p(\delta')) = c_\delta(p) = c_\delta(p') = c_{\mathbb{A}}(p' \restriction \delta', p'(\delta')).$$

  So, as $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}_{\delta'}, \ell_{\delta'}, c_{\delta'})$, we have $c_{\delta'}(p \restriction \delta') = c_{\delta'}(p' \restriction \delta')$, and, for all $r \in (P_{\delta'})_0^{p\restriction\delta'} \cap (P_{\delta'})_0^{p'\restriction\delta'}$, $\pitchfork(p \restriction \delta', p(\delta'))(r) = \pitchfork(p' \restriction \delta', p'(\delta'))(r)$.

  Now, as $c_{\delta'}(p \restriction \delta') = c_{\delta'}(p' \restriction \delta')$, the induction hypothesis implies that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all nonzero $\gamma \leq \delta'$. In addition, the case $\gamma = \delta$ is trivial.

  Finally, fix a nonzero $\gamma \leq \delta$ and $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{p'\restriction\gamma}$, and let us prove that $\pitchfork_{\delta,\gamma}(p)(r) = \pitchfork_{\delta,\gamma}(p')(r)$. To avoid trivialities, assume $\gamma < \delta$. It follows from the definition of $\pitchfork_{\delta,\gamma}$ that $\pitchfork_{\delta,\gamma}(p)(r) = x^\frown\langle y\rangle$ and $\pitchfork_{\delta,\gamma}(p')(r) = x'^\frown\langle y'\rangle$, where:
  - $(x, y) := \pitchfork(p \restriction \delta', p(\delta'))(\pitchfork_{\delta',\gamma}(p \restriction \delta')(r))$, and
  - $(x', y') := \pitchfork(p' \restriction \delta', p'(\delta'))(\pitchfork_{\delta',\gamma}(p' \restriction \delta')(r))$.

  But we have already pointed out that the induction hypothesis implies that $\pitchfork_{\delta',\gamma}(p\restriction\delta')(r) = \pitchfork_{\delta',\gamma}(p'\restriction\delta')(r)$, call it, r'. So, we just need to prove that $\pitchfork(p \restriction \delta', p(\delta'))(r') = \pitchfork(p' \restriction \delta', p'(\delta'))(r')$. But we also have $c_{\mathbb{A}}(p \restriction \delta, p(\delta')) = c_\delta(p) = c_\delta(p') = c_{\mathbb{A}}(p' \restriction \delta, p'(\delta'))$ and by our choice of r and Clause (2) of Claim 3.6.1, $r' \in (P_{\delta'})_0^{p\restriction\delta'} \cap (P_{\delta'})_0^{p'\restriction\delta'}$. So, as $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}_{\delta'}, \ell_{\delta'}, c_{\delta'})$, Clause (8) of Definition 2.13 implies that

  $$\pitchfork(p \restriction \delta', p(\delta'))(r') = \pitchfork(p' \restriction \delta', p'(\delta'))(r'),$$

  as desired.
- ▶ Suppose $\delta \in \mathrm{acc}(\mu^+)$ is an ordinal for which the claim holds below δ. For any condition $q \in \bigcup_{\delta' \leq \delta} P_{\delta'}$, define a function $f_q : B_q \to H_\mu$ via $f_q(\delta') := c_{\delta'}(q \restriction \delta')$. Now, fix an arbitrary pair $p, p' \in P_\delta$ with $c_\delta(p) = c_\delta(p')$. By the definition of c_δ this means that

  $$\{(\phi_\delta(\gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\} = \{(\phi_\delta(\gamma), c_\gamma(p' \restriction \gamma)) \mid \gamma \in B_{p'}\}.$$

  As ϕ_δ is injective, $f_p = f_{p'}$. Next, let $\gamma \leq \delta$ be nonzero; we need to show that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$. The case $\gamma = \delta$ is trivial, so assume $\gamma < \delta$.

Now, if $\mathrm{dom}(f_p) \setminus \gamma$ is nonempty, then for $\delta' := \min(\mathrm{dom}(f_p) \setminus \gamma)$, we have $c_{\delta'}(p \restriction \delta') = f_p(\delta') = f_{p'}(\delta') = c_{\delta'}(p' \restriction \delta')$, and then the induction hypothesis entails $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$. In particular, if $\mathrm{dom}(f_p)$ is unbounded in δ, then $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all $\gamma \leq \delta$.

Next, suppose that $\mathrm{dom}(f_p)$ is bounded in δ and let $\epsilon < \delta$ be the least ordinal to satisfy $\mathrm{dom}(f_p) \subseteq \epsilon$. We already know that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all $\gamma < \epsilon$, and now prove by induction that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$ for all $\gamma \in [\epsilon, \delta)$. For a successor ordinal γ, this follows from Clauses (f) and (g) of Building Block II, and for a limit ordinal γ, this follows from the fact that the injectivity of ϕ_γ and the equality $f_{p\restriction\gamma} = f_p = f_{p'} = f_{p'\restriction\gamma}$ implies that $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$.

Finally, fix a nonzero $\gamma \leq \delta$ and $r \in (P_\gamma)_0^{p\restriction\gamma} \cap (P_\gamma)_0^{p'\restriction\gamma}$, and let us prove that $\pitchfork_{\delta,\gamma}(p)(r) = \pitchfork_{\delta,\gamma}(p')(r)$. To avoid trivialities, assume $\gamma < \delta$. We already know that, for all $\delta' \in [\gamma, \delta)$, $c_{\delta'}(p \restriction \delta') = c_{\delta'}(p' \restriction \delta')$, and so the induction hypothesis implies that $\pitchfork_{\delta',\gamma}(p \restriction \delta')(r) = \pitchfork_{\delta',\gamma}(p' \restriction \delta')(r)$, and then by Lemma 3.5(1):

$$\begin{aligned}\pitchfork_{\delta,\gamma}(p)(r) &= \bigcup_{\gamma\leq\delta'<\delta} \pitchfork_{\delta',\gamma}(p \restriction \delta')(r) = \\ &= \bigcup_{\gamma\leq\delta'<\delta} \pitchfork_{\delta',\gamma}(p' \restriction \delta')(r) = \pitchfork_{\delta,\gamma}(p')(r),\end{aligned}$$

as desired. □

Claim 3.6.3. *If $\delta = \beta + 1 > 1$, then $(\pitchfork_{\delta,\beta}, \pi_{\delta,\beta})$ has the weak mixing property.*

Proof. Once again, recall that $\mathbb{P}_{\beta+1}$ was obtained by feeding $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ into Building Block II, thus obtaining a Σ-Prikry triple $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$, along with a pair $(\pitchfork, \pi)$ having the weak mixing property. Let tp be a type over $(\pitchfork, \pi)$ witnessing the weak mixing property. For each condition $p \in P_{\beta+1}$, set $\mathsf{tp}_{\beta+1}(p) := \mathsf{tp}(p \restriction \beta, p(\beta))$. The canonical isomorphism from $\mathbb{A}$ to $\mathbb{P}_{\beta+1}$ (i.e., $(x, y) \mapsto x^\frown\langle y\rangle$) makes it clear that $\mathsf{tp}_{\beta+1}$ is a type over $(\pitchfork_{\beta+1,\beta}, \pi_{\beta+1,\beta})$ witnessing the weak mixing property. □

This completes the proof of Lemma 3.6. □

Definition 3.7. For each nonzero $\beta < \mu^+$, we let $\mathsf{tp}_{\beta+1}$ be the witness to the weak mixing property of $(\pitchfork_{\beta+1,\beta}, \pi_{\beta+1,\beta})$, as defined in the proof of Subclaim 3.6.3.

Recalling Definition 2.3(2), for all nonzero $\delta \leq \mu^+$ and $n < \omega$, we need to identify a candidate for a dense subposet $\mathring{\mathbb{P}}_{\delta n} = (\mathring{P}_{\delta n}, \leq_\delta)$ of $\mathbb{P}_{\delta n}$. We do this next.

Definition 3.8. Let $n<\omega$. Set $\mathring{P}_{1n}:={}^1(\mathring{Q}_n)$.[11] Then, for each $\delta\in[2,\mu^+]$, define $\mathring{P}_{\delta n}$ by recursion:

$$\mathring{P}_{\delta n}:=\begin{cases}\{p\in P_\delta\mid \pi_{\delta,\beta}(p)\in\mathring{P}_{\beta n}\ \&\ \mathrm{mtp}_{\beta+1}(p)=0\}, & \text{if } \delta=\beta+1;\\ \{p\in P_\delta\mid \pi_{\delta,1}(p)\in\mathring{P}_{1n}\ \&\ \forall\gamma\in B_p\ \mathrm{mtp}_\gamma(\pi_{\delta,\gamma}(p))=0\}, & \text{otherwise.}\end{cases}$$

Lemma 3.9. *Let $n<\omega$ and $1\leq\beta<\delta\leq\mu^+$. Then:*

(1) $\pi_{\delta,\beta}$"$\mathring{P}_{\delta n}\subseteq\mathring{P}_{\beta n}$;
(2) For every $p\in\mathring{P}_{\beta n}$, $p*\emptyset_\delta\in\mathring{P}_{\delta n}$.

Proof. By a straight-forward induction, relying on Clause (4) of Definition 2.23. □

We are now left with addressing Clause (iv) of Goal 3.2. Prior to that we will provide a sufficient condition securing that for each $\delta\in\mathrm{acc}(\mu^+{+}1)$, the pair $(\mathbb{P}_\delta,\ell_\delta)$ has property $\mathcal{D}$. For this, we establish a version of the Weak Mixing Property (Definition 2.25) for limit stages.

Lemma 3.10. *Let $\delta\in\mathrm{acc}(\mu^+{+}1)$. For all $a\in P_\delta$, $n<\omega$, $\vec{r}$, and $p'\leq^0\pi_{\delta,1}(a)$, and for every function $g:W_n(\pi_{\delta,1}(a))\to\mathbb{P}_\delta\downarrow a$, if all of the following hold:*

(0) $\langle B_{g(r_\xi)}\mid\xi<\chi\rangle$ is $\subseteq$-weakly increasing. For each γ in $B:=\bigcup_{\xi<\chi}B_{g(r_\xi)}$, we denote $\iota_\gamma:=\min\{\xi<\chi\mid\gamma\in B_{g(r_\xi)}\}$;
(1) $\vec{r}=\langle r_\xi\mid\xi<\chi\rangle$ is a good enumeration of $W_n(\pi_{\delta,1}(a))$;
(2) $\langle\pi_{\delta,1}(g(r_\xi))\mid\xi<\chi\rangle$ is diagonalizable with respect to $\vec{r}$ as witnessed by p';
(3) for all $\gamma\in B$ and $\xi<\chi$,

$$\begin{cases}\mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\xi))))=0, & \xi<\iota_\gamma;\\ \mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\xi))))\geq 1, & \xi=\iota_\gamma;\\ \mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\xi))))>(\sup_{\eta<\xi}\mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\eta)))))+1, & \xi>\iota_\gamma.\end{cases}$$

(4) for all $\gamma\in B$, $\xi\in(\iota_\gamma,\chi)$, and $i\leq\sup_{\eta<\xi}\mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\eta))))$,

$$i\geq\mathrm{dom}(\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(a)))\implies\mathsf{tp}_\gamma(\pi_{\delta,\gamma}(g(r_\xi)))(i)\leq\mathrm{mtp}_\gamma(\pi_{\delta,\gamma}(a));$$

(5) for all $\gamma\in B$, $\sup_{\iota_\gamma\leq\xi<\chi}\mathrm{mtp}_\gamma(\pi_{\delta,\gamma}(g(r_\xi)))<\omega$,

then there exists $b\in P_\delta$ such that:

(a) $\pi_{\delta,1}(b)=p'$;
(b) $b\leq^0_\delta a$;
(c) for all $q'\in W_n(p')$, $\Cap_{\delta,1}(b)(q')\leq^0_\delta g(w(\pi_{\delta,1}(a),q'))$.

Proof. Let $a\in P_\delta$, $n<\omega$, $\vec{r}$, p' and $g:W_n(a\restriction 1)\to\mathbb{P}_\delta\downarrow a$ be as above. Let $\langle\gamma_\tau\mid\tau<\theta\rangle$ be the increasing enumeration of B. From Goal 3.2(i) and $\chi<\mu$, we infer that $\theta<\mu$. For each $\tau<\theta$:

- as γ_τ is a successor ordinal, we let β_τ denote its predecessor;

[11] Here, $\mathring{Q}_n$ is obtained from Clause (2) of Definition 2.3 with respect to the triple $(\mathbb{Q},\ell,c)$ given by Building Block I.

- for every $\xi<\chi$, let $r^\tau_\xi := \Cap_{\beta_\tau,1}(a\restriction\beta_\tau)(r_\xi)$. By Fact 2.19, $\vec{r}^\tau := \langle r^\tau_\xi \mid \xi<\chi\rangle$ is a good enumeration of $W_n(a\restriction\beta_\tau)$;
- derive a map $g_\tau\colon W_n(a\restriction\beta_\tau)\to\mathbb{P}_{\gamma_\tau}\downarrow(a\restriction\gamma_\tau)$ via

$$g_\tau(r^\tau_\xi) := g(r_\xi)\restriction\gamma_\tau.$$

Claim 3.10.1. *Suppose there is a sequence $\langle(b_\tau,p^\tau)\mid\tau<\theta\rangle\in\prod_{\tau<\theta}(P_{\gamma_\tau}\times P_{\beta_\tau})$ satisfying that for all $\tau<\theta$:*

(I) $b_0\restriction 1 = p^0\restriction 1 = p'$;
(II) $b_\tau\restriction\gamma_{\tau'} = b_{\tau'}$ *for all* $\tau'<\tau$;
(III) b_τ *witnesses the conclusion of Definition 2.25 with respect to the tuple* $(a\restriction\gamma_\tau,\vec{r}^\tau,p^\tau,g_\tau,\iota_{\gamma_\tau})$. *In particular,* $p^\tau\leq^0_{\beta_\tau} a\restriction\beta_\tau$ *diagonalizes* $\langle g_\tau(r^\tau_\xi)\restriction\beta_\tau\mid\xi<\chi\rangle$.

Then there is $b\in P_\delta$ as in the conclusion of the Lemma.

Proof. By (II) above, we may let $b^*:=\bigcup_{\tau<\theta}b_\tau$, so that $b^*\in P_\epsilon$ for $\epsilon:=\mathrm{dom}(b^*)$. For each $\tau<\theta$, Clauses (II) and (III) yield

$$b^*\restriction\gamma_\tau = b_\tau\leq^0_{\gamma_\tau} a\restriction\gamma_\tau,$$

and hence $b^*\leq^0_\epsilon(a\restriction\epsilon)$. So we may let $b:=\Cap_{\delta,\epsilon}(a)(b^*)$, and infer from (I) that $b\restriction 1=p'$. Also, we have that $b\restriction\gamma\leq^0_\gamma a\restriction\gamma$, for each $\gamma\in B_a$. This shows that Clauses (a) and (b) of the lemma hold.

We are now left with verifying Clause (c). Let $q'\in W_n(p')$; we want to show that $\Cap_{\delta,1}(b)(q')\leq^0_\delta g(w(a\restriction 1,q'))$. Note that by Lemma 3.5(2), $B_a\subseteq B_{b^*}=B_b$, so that $b=b^**\emptyset_\delta$. Hence, $\{\gamma_\tau\mid\tau<\theta\}$ is cofinal in B_b, and so it suffices to prove that, for each $\tau<\theta$,

$$\Cap_{\delta,1}(b)(q')\restriction\gamma_\tau\leq^0_{\gamma_\tau} g(w(a\restriction 1,q'))\restriction\gamma_\tau.$$

For each $\tau<\theta$, combining Clause (II) with Lemma 3.5(1) we have

$$\Cap_{\delta,1}(b)(q')\restriction\gamma_\tau = \Cap_{\gamma_\tau,1}(b\restriction\gamma_\tau)(q') = \Cap_{\gamma_\tau,1}(b_\tau)(q'),$$

hence it suffices to check that

$$(\star)\qquad \Cap_{\gamma_\tau,1}(b_\tau)(q')\leq^0_{\gamma_\tau} g(w(a\restriction 1,q'))\restriction\gamma_\tau.$$

By the definition of $\Cap_{\gamma_\tau,1}$ (see page 16, especially equation (*)) it follows

$$(\star\star)\qquad \Cap_{\gamma_\tau,1}(b_\tau)(q') = \Cap_{\gamma_\tau,\beta_\tau}(b_\tau)(\Cap_{\beta_\tau,1}(b_\tau\restriction\beta_\tau)(q')).$$

Since $b_\tau\restriction 1=p'$ and $q'\in W_n(p')$, Lemma 3.6 yields $r:=\Cap_{\beta_\tau,1}(b_\tau\restriction\beta_\tau)(q')$ is in $W_n(b_\tau\restriction\beta_\tau)$. Combining equation $(\star\star)$ with (III), we infer that

$$\Cap_{\gamma_\tau,1}(b_\tau)(q') = \Cap_{\gamma_\tau,\beta_\tau}(b_\tau)(r)\leq^0_{\gamma_\tau} g_\tau(w(a\restriction\beta_\tau,r)) = g(w(a\restriction 1,q'))\restriction\gamma_\tau,$$

where the rightmost equality follows from the definition of g_τ and the fact that $r\restriction 1=q'$. This verifies equation $(\star)$ and yields the claim. □

Let us now argue by induction that such $\langle(b_\tau,p^\tau)\mid\tau<\theta\rangle$ exists.

Claim 3.10.2. *There is a pair (b_0,p^0) for which Clauses* (I)–(III) hold.

Proof. Clause (II) is trivial at this stage. Setting $p^0 := \Cap_{\beta_0,1}(a \restriction \beta_0)(p')$ takes care of the second part of Clause (I), and we shall come back to the first part towards the end. Now, let us examine the tuple $(a \restriction \gamma_0, \vec{r}^0, p^0, g_0, \iota_{\gamma_0})$ against the clauses of Definition 2.25 with respect to the forking projection $(\Cap_{\gamma_0,\beta_0}, \pi_{\gamma_0,\beta_0})$: Clause (1) is obvious and Clauses (3), (4) and (5) follow combining the corresponding clauses in the lemma with the definition of g_0.

Regarding Clause (2), we claim that p^0 diagonalizes $\langle g_0(r^0_\xi) \restriction \beta_0 \mid \xi < \chi\rangle$.

To that effect we will check (α) and (β) of Proposition 2.22, when this is regarded with respect to the forking projection $(\Cap_{\beta_0,1}, \pi_{\beta_0,1})$, and the parameters $a \restriction \beta_0$, $\vec{r}^0$, $\langle g_0(r^0_\xi) \restriction \beta_0 \mid \xi < \chi\rangle$, p' and p^0, respectively.

- (α) Note that $g_0(r^0_\xi) \restriction 1 = g(r_\xi) \restriction 1$, for each $\xi < \chi$. Therefore, Clause (1) implies that p' diagonalizes $\langle g_0(r^0_\xi) \restriction 1 \mid \xi < \chi\rangle$.
- (β) Note that by Clause (1) of the lemma, $p' \leq^0_1 a \restriction 1$, hence $p^0 \leq^0_{\beta_0} a \restriction \beta_0$.

  Let $q' \in W_n(p')$. Again by Clause (1), $q' \leq^0_1 g(r_\xi) \restriction 1$, where ξ is the unique index such that $r_\xi = w(a \restriction 1, q' \restriction 1)$.

  Finally, combining Lemma 3.5(5) and Lemma 3.6 we have

$$\Cap_{\beta_0,1}(p^0)(q') \leq^0_{\beta_0} \Cap_{\beta_0,1}(a \restriction \beta_0)(g(r_\xi) \restriction 1) = g(r_\xi) \restriction 1 * \emptyset_{\beta_0} = g_0(r^0_\xi) \restriction \beta_0,$$

  where the above equalities follow from $\beta_0 < \min(B)$.

Altogether, $(a \restriction \gamma_0, \vec{r}^0, p^0, g_0, \iota_{\gamma_0})$ witnesses Clauses (1)–(5) of Definition 2.25. Thus, appealing to Lemma 3.6, we obtain $b \in P_{\gamma_0}$ such that $b \restriction \beta_0 = p^0$ and $b \leq^0_{\gamma_0} a \restriction \gamma_0$ that witnesses the conclusion of Definition 2.25. Clearly, $b_0 := b$ and p^0 are as wanted. □

Suppose now $\tau < \theta$, and that $\langle (b_{\tau'}, p^{\tau'}) \mid \tau' < \tau\rangle$ has been constructed maintaining (I)–(III). Set $b^* := \bigcup_{\tau'<\tau} b_{\tau'}$ and $\epsilon := \mathrm{dom}(b^*)$. Note that $\epsilon \leq \beta_\tau$, as $\gamma_\tau \in \mathrm{nacc}(\mu^+)$. Also, using (I) and (II) of the induction, $b^* \in P_\epsilon$ and $\pi_{\epsilon,1}(b^*) = p'$.

Claim 3.10.3. *There is a pair (b_τ, p^τ) satisfying Clauses* (I)–(III).

Proof. We commence checking (III). As in the previous claim, it suffices to show that $p^\tau := \Cap_{\beta_\tau,\epsilon}(a \restriction \beta_\tau)(b^*)$ diagonalizes $\langle g_\tau(r^\tau_\xi) \restriction \beta_\tau \mid \xi < \chi\rangle$.

Once again, we want to appeal to Proposition 2.22, but this time regarded with respect to $(\Cap_{\beta_\tau,1}\, \pi_{\beta_\tau,1})$, $a \restriction \beta_\tau$, $\vec{r}^\tau$, $\langle g_\tau(r^\tau_\xi) \restriction \beta_\tau \mid \xi < \chi\rangle$, p' and p^τ. The verification of Clause (α) is exactly the same as in Claim 3.10.2, so we focus on Clause (β). By (II) and (III) of the induction hypothesis, $b^* \leq^0_\epsilon a \restriction \epsilon$ and $b^* \restriction 1 = p'$. Hence, $p^\tau \in P_{\beta_\tau}$, $p^\tau \leq^0_{\beta_\tau} a \restriction \beta_\tau$ and $p^\tau \restriction 1 = p'$.

Let $q' \in W_n(p')$. Our aim is to show that

$$\Cap_{\beta_\tau,1}(p^\tau)(q') \leq^0_{\beta_\tau} g_\tau(r^\tau_\xi) \restriction \beta_\tau,$$

for the unique index ξ such that $r_\xi = w(a \restriction 1, q')$.

By virtue of Lemma 3.5(5), $B_{\Cap_{\beta_\tau,1}(p_\tau)(q')} = B_{p^\tau} = B_{b^*}$. Hence, it will be enough to check that $\Cap_{\beta_\tau,1}(p^\tau)(q') \restriction \epsilon \leq^0_\epsilon g_\tau(r^\tau_\xi) \restriction \epsilon$.

For each $\tau' < \tau$, combining (II) of the induction hypothesis with Clauses (1) and (3) of Lemma 3.5 we have

$$\Cap_{\beta_\tau,1}(p^\tau)(q') \restriction \gamma_{\tau'} = \Cap_{\gamma_{\tau'},1}(b_{\tau'})(q') = \Cap_{\gamma_{\tau'},\beta_{\tau'}}(b_{\tau'})(s_{\tau'}),$$

where $s_{\tau'} := \Cap_{\beta_{\tau'},1}(b_{\tau'} \restriction \beta_{\tau'})(q')$.[12]

Thus, by (III) of our induction hypothesis,

$$\Cap_{\beta_\tau,1}(p^\tau)(q') \restriction \gamma_{\tau'} = \Cap_{\gamma_{\tau'},\beta_{\tau'}}(b_{\tau'})(s_{\tau'}) \leq^0_{\gamma_{\tau'}} g_{\tau'}(r^{\tau'}_\xi),$$

where ξ is the unique index such that $r^{\tau'}_\xi = w(a \restriction \beta_{\tau'}, s_{\tau'})$.

Since $g_\tau(r^\tau_\xi) \restriction \gamma_{\tau'} = g_{\tau'}(r^{\tau'}_\xi)$, the above expression actually yields $\Cap_{\beta_\tau,1}(p^\tau)(q') \restriction \gamma_{\tau'} \leq^0_{\gamma'_\tau} g_\tau(r^\tau_\xi) \restriction \gamma_{\tau'}$. Altogether, we have shown that

$$\Cap_{\beta_\tau,1}(p^\tau)(q') \restriction \epsilon \leq^0_\epsilon g_\tau(r^\tau_\xi) \restriction \epsilon.$$

Finally, note that

$$r_\xi = r^{\tau'}_\xi \restriction 1 = w(a \restriction \beta_{\tau'}, s_{\tau'}) \restriction 1 = w(a \restriction 1, q'),$$

where the last equality follows from Lemma 3.6 and $s_{\tau'} \restriction 1 = q'$.

The above shows that $(a \restriction \gamma_\tau, \vec{r}^\tau, p^\tau, g_\tau, i_{\gamma_\tau})$ fulfills the assumptions of Definition 2.25 with respect the pair $(\Cap_{\gamma_\tau,\beta_\tau}, \pi_{\gamma_\tau,\beta_\tau})$. Appealing to Lemma 3.6 we obtain $b_\tau \leq^0_{\gamma_\tau} a \restriction \gamma_\tau$ with $b_\tau \restriction \beta_\tau = p^\tau$ such that the pair (b_τ, p^τ) witnesses (III).

Let us now show that (b_τ, p^τ) satisfies (I) and (II). By (II) of the induction hypothesis and the definition of p^τ, for each $\tau' < \tau$,

$$b_\tau \restriction \gamma_{\tau'} = (p^\tau \restriction \epsilon) \restriction \gamma_{\tau'} = b^* \restriction \gamma_{\tau'} = b_{\tau'}.$$

Similarly, by (I) of the induction hypothesis, $b_\tau \restriction 1 = b_{\tau'} \restriction 1 = p'$. □

The above completes the induction and yields the lemma. □

The following technical lemma yields a sufficient condition for the pair $(\mathbb{P}_\delta, \ell_\delta)$ to have property $\mathcal{D}$ in case $\delta \in \mathrm{acc}(\mu^+ + 1)$.

Lemma 3.11. *Let $\delta \in \mathrm{acc}(\mu^+ + 1)$, $a \in P_\delta$, $n < \omega$ and $\vec{s} = \langle s_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$. Set $l := \ell_\delta(a)$. Suppose that D is a set of conditions in $(\mathring{\mathbb{P}}_\delta)_{l+n}$ which is dense in $(\mathbb{P}_\delta)_{l+n}$. Then* **I** *has a winning strategy for the game $\Game_{\mathbb{P}_\delta}(a, \vec{s}, D)$.*

Proof. Set $p := \pi_{\delta,1}(a)$ and $r_\xi := \pi_{\delta,1}(s_\xi)$ for each $\xi < \chi$. By Clauses (4) and (5) of Definition 2.13, $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of $W_n(p)$.

We now describe our strategy for **I**. Suppose that $\xi < \chi$ and that $\langle (a_\eta, b_\eta) \mid \eta < \xi \rangle$ is an initial play of the game $\Game_{\mathbb{P}_\delta}(a, \vec{s}, D)$; we need to define a_ξ.

▶ If $\xi = 0$, then let p_0 be the 0^{th}-move of **I** according to some winning strategy in $\Game_{\mathbb{P}_1}(p, \vec{r})$, which is available by virtue of Building Block I. Also, let t_0 be a condition in $\mathbb{P}_1$ such that $t_0 \leq_1 p_0, r_0$.[13]

[12] For this latter equality, see equation ($\star\star$) above.

[13] Recall that this is part of the rules of $\Game_{\mathbb{P}_1}(p, \vec{r})$ (see Definition 2.10).

If B_a is empty, then let $a_0 := \Cap_{\delta,1}(a)(p_0)$ and $z_0 := \Cap_{\delta,1}(a)(t_0)$. By Lemma 3.6 $z_0 \leq_\delta a_0, s_0$, hence a_0 is a legitimate move for **I**.

Suppose now that B_a is nonempty, and let $\langle \gamma_\tau \mid \tau \leq \theta \rangle$ be the increasing enumeration of the closure of B_a. For every $\tau \in \mathrm{nacc}(\theta+1)$, γ_τ is a successor ordinal, so we let β_τ denote its predecessor. By recursion on $\tau \leq \theta$, we shall define a coherent sequence[14] $\langle (a_0^\tau, z_0^\tau) \mid \tau \leq \theta \rangle \in \prod_{\tau \leq \theta}(P_{\gamma_\tau} \times P_{\gamma_\tau})$, and then we shall let $a_0 := \Cap_{\delta,\gamma_\theta}(a)(a_0^\theta)$ and $z_0 := \Cap_{\delta,\gamma_\theta}(a)(z_0^\theta)$.

The idea is to craft the sequence $\langle a_0^\tau \mid \tau \leq \theta \rangle$ so that for all $\gamma \in B_a$, $a_0 \restriction \gamma$ satisfies (2)–(4) of Lemma 3.10. On the other hand, $\langle z_0^\tau \mid \tau \leq \theta \rangle$ will provide a sequence of auxiliary conditions witnessing that $z_0^\tau \leq_{\gamma_\tau} a_0^\tau, s_0 \restriction \gamma_\tau$. This will ensure at the end that a_0 is a legitimate move for **I**.

►► Set $\varrho_0^0 := \mathrm{dom}(\mathtt{tp}_{\gamma_0}(a \restriction \gamma_0)) + \omega + 1$, and then let

$$a_0^0 := \Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0)^\frown \varrho_0^0,$$
$$z_0^0 := \Cap_{\gamma_0,1}(a \restriction \gamma_0)(t_0)^\frown \varrho_0^0,$$

where the $^\frown$ operation is provided by Definition 2.23(5) with respect to the type $\mathtt{tp}_{\gamma_0}$ over the forking projection $(\Cap_{\gamma_0,\beta_0}, \pi_{\gamma_0,\beta_0})$.[15]

Since $p_0 \leq_1^0 a \restriction 1$, $a_0^0 \in P_{\gamma_0}$ and also $a_0^0 \leq_{\gamma_0}^0 a \restriction \gamma_0$. Similarly, $z_0^0 \in P_{\gamma_0}$.

Claim 3.11.1. $z_0^0 \leq_{\gamma_0} a_0^0, s_0 \restriction \gamma_0$.

Proof. Combining Clause (5)(a) of Definition 2.23 with Lemma 3.6,

$$z_0^0 \leq_{\gamma_0}^0 \Cap_{\gamma_0,1}(a \restriction \gamma_0)(t_0) \leq_{\gamma_0}^0 \Cap_{\gamma_0,1}(a \restriction \gamma_0)(r_0) = s_0 \restriction \gamma_0.$$

On the other hand, $\Cap_{\gamma_0,1}(a \restriction \gamma_0)(t_0) \leq_{\gamma_0} \Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0)$ and

$$\mathrm{dom}(\mathtt{tp}_{\gamma_0}(\Cap_{\gamma_0,1}(a \restriction \gamma_0)(t_0))) = \mathrm{dom}(\mathtt{tp}_{\gamma_0}(\Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0))),$$

where this last equality follows from Clause (3) of Definition 2.23.

Combining this with Definition 2.23(6) we get $z_0^0 \leq_{\gamma_0} a_0^0$, as desired. □

Claim 3.11.2. *For all* $i \in [\mathrm{dom}(\mathtt{tp}_{\gamma_0}(a \restriction \gamma_0)), \mathrm{dom}(\mathtt{tp}_{\gamma_0}(a_0^0))]$,

$$\mathtt{tp}_{\gamma_0}(a_0^0)(i) \leq \mathrm{mtp}_{\gamma_0}(a \restriction \gamma_0).$$

Proof. Let i be as above. By Definition 2.23(3), $\mathrm{dom}(\mathtt{tp}_{\gamma_0}(\Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0)) = \mathrm{dom}(\mathtt{tp}_{\gamma_0}(a \restriction \gamma_0))$. So, combining Clauses (2) and (5) of Definition 2.23

$$\mathtt{tp}_{\gamma_0}(a_0^0)(i) \leq \mathrm{mtp}_{\gamma_0}(\Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0)) \leq \mathrm{mtp}_{\gamma_0}(a \restriction \gamma_0),$$

as desired. □

►► For every $\tau < \theta$ such that both a_0^τ and z_0^τ have already been defined, set $\varrho_0^{\tau+1} := \mathrm{dom}(\mathtt{tp}_{\gamma_{\tau+1}}(a \restriction \gamma_{\tau+1})) + \omega + 1$, and then let

$$a_0^{\tau+1} := \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(a_0^\tau)^\frown \varrho_0^{\tau+1},$$
$$z_0^{\tau+1} := \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(z_0^\tau)^\frown \varrho_0^{\tau+1}.$$

[14] Namely, for each $\tau' \leq \tau$, $a_0^\tau \restriction \tau' = a_0^{\tau'}$ and $z_0^\tau \restriction \tau' = z_0^{\tau'}$.

[15] Note that $\Cap_{\gamma_0,1}(a \restriction \gamma_0)(p_0) = \Cap_{\gamma_0,\beta_0}(a \restriction \gamma_0)(\Cap_{\beta_0,1}(a \restriction \beta_0)(p_0))$ (see (*) on page 16).

where the $\curvearrowright$ operation is with respect to the type $\mathsf{tp}_{\gamma_{\tau+1}}$.[16]

Claim 3.11.3. *For all $\tau' \leq \tau$, $a_0^{\tau+1} \restriction \gamma_{\tau'} = a_0^{\tau'}$ and $z_0^{\tau+1} \restriction \gamma_{\tau'} = z_0^{\tau'}$.*

Proof. Let $\tau' \leq \tau$. By Clause (5)(a) of Definition 2.23 and Lemma 3.5(1),

$$a_0^{\tau+1} \restriction \beta_{\tau+1} = \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(a_0^\tau) \restriction \beta_{\tau+1} = \Cap_{\beta_{\tau+1},\gamma_\tau}(a \restriction \beta_{\tau+1})(a_0^\tau).$$

Hence, Lemma 3.5(5) yields $a_0^{\tau+1} \restriction \gamma_\tau = a_0^\tau$. Using the induction hypothesis, we get $a_0^{\tau+1} \restriction \gamma_{\tau'} = a_0^{\tau'}$. The argument for $z_0^{\tau+1}$ is the same. □

Claim 3.11.4. $z_0^{\tau+1} \leq_{\gamma_{\tau+1}} a_0^{\tau+1}, s_0 \restriction \gamma_{\tau+1}$.

Proof. Recall that by the induction hypothesis, $z_0^\tau \leq_{\gamma_\tau} a_0^\tau, s_0 \restriction \gamma_\tau$. Thus, Clause (5) of Definition 2.23 and Lemma 3.6 combined yield

$$z_0^{\tau+1} \leq^0_{\gamma_{\tau+1}} \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(z_0^\tau) \leq^0_{\gamma_{\tau+1}} \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(s_0 \restriction \gamma_\tau) = s_0 \restriction \gamma_{\tau+1}.$$

Similarly, Lemma 3.6 yields

$$\Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(z_0^\tau) \leq_{\gamma_{\tau+1}} \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(a_0^\tau).$$

Also, by Clause (3) of Definition 2.23 and the remark made at Footnote 16

$$\mathrm{dom}(\mathsf{tp}_{\gamma_{\tau+1}}(\Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(z_0^\tau))) = \mathrm{dom}(\mathsf{tp}_{\gamma_{\tau+1}}(\Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(a_0^\tau)).$$

Therefore, Definition 2.23(6) yields $z_0^{\tau+1} \leq_{\gamma_{\tau+1}} a_0^{\tau+1}$, as desired. □

Finally, the following can be proved exactly as in Claim 3.11.2:

Claim 3.11.5. *For all $i \in [\mathrm{dom}(\mathsf{tp}_{\gamma_{\tau+1}}(a \restriction \gamma_{\tau+1})), \mathrm{dom}(\mathsf{tp}_{\gamma_{\tau+1}}(a_0^{\tau+1}))]$,*

$$\mathsf{tp}_{\gamma_{\tau+1}}(a_0^{\tau+1})(i) \leq \mathrm{mtp}_{\gamma_{\tau+1}}(a \restriction \gamma_{\tau+1}).$$

►► For every $\tau \in \mathrm{acc}(\theta+1)$, let $a_0^\tau := \bigcup_{\tau'<\tau} a_0^{\tau'}$ and $z_0^\tau := \bigcup_{\tau'<\tau} z_0^{\tau'}$. Thanks to the induction hypothesis, $\langle (a_0^{\tau'}, z_0^{\tau'}) \mid \tau' \leq \tau \rangle$ is clearly coherent. Clearly, $z_0^\tau \leq_{\gamma_\tau} a_0^\tau$ and arguing as in Claim 3.11.4 we have $z_0^\tau \leq^0_{\gamma_\tau} s_0 \restriction \gamma_\tau$.

At the end of the recursion, we define a_0 and z_0 as mentioned before. Note that by our construction z_0 witnesses that a_0 is a legitimate move for **I** so, in response, **II** plays a condition b_0 in D extending a_0 and satisfying $b_0 \leq^0_\delta s_0$. Finally, note that $B_a \subseteq B_{a_0} \subseteq B_{b_0}$[17] and for every $\gamma \in B_a$,

$$\mathrm{dom}(\mathsf{tp}_\gamma(a \restriction \gamma)) + 1 < \mathrm{dom}(\mathsf{tp}_\gamma(a_0 \restriction \gamma)).$$

Also, for all $i \in [\mathrm{dom}(\mathsf{tp}_\gamma(a \restriction \gamma)), \mathrm{dom}(\mathsf{tp}_\gamma(a_0 \restriction \gamma))]$,

$$\mathsf{tp}_\gamma(a_0 \restriction \gamma)(i) \leq \mathrm{mtp}_\gamma(a \restriction \gamma).$$

► Suppose that $0 < \xi < \chi$. Recall that $\langle (a_\eta, b_\eta) \mid \eta < \xi \rangle$ is an initial play of the game and that we want to define a_ξ. To that effect, let p_ξ the ξ^{th}-move of **I** in the game $\Game_{\mathbb{P}_1}(p, \vec{r})$, provided $\langle (a_\eta \restriction 1, b_\eta \restriction 1) \mid \eta < \xi \rangle$ gathers the previous ones. Let t_ξ be such that $t_\xi \leq_1 p_\xi, s_\xi$ and set $B_\xi := \bigcup_{\eta<\xi} B_{b_\eta}$.

[16]Note that $\Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(a_0^\tau) = \Cap_{\gamma_{\tau+1},\beta_{\tau+1}}(a \restriction \gamma_{\tau+1})(\Cap_{\beta_{\tau+1},\gamma_\tau}(a \restriction \beta_{\tau+1})(a_0^\tau))$.

[17]The inclusion $B_{a_0} \subseteq B_{b_0}$ is obvious. For the other we use Clause (4) of Definition 2.23, noting that for all $\gamma \in B_a$, $\mathrm{dom}(\mathsf{tp}_\gamma(a_0 \restriction \gamma)) \neq 0$.

If B_ξ is empty then again set $a_\xi := \Cap_{\delta,1}(a)(p_\xi)$ and $z_\xi := \Cap_{\delta,1}(a)(t_\xi)$ and argue as in the case $\xi = 0$. Otherwise, B_ξ is nonempty and we let $\langle \gamma_\tau \mid \tau \leq \theta\rangle$ be the increasing enumeration of the closure of B_ξ. By recursion on $\tau \leq \theta$, we define a coherent sequence $\langle (a^\tau_\xi, z^\tau_\xi) \mid \tau \leq \theta\rangle \in \prod_{\tau\leq\theta}(P_{\gamma_\tau} \times P_{\gamma_\tau})$, and then we shall let $a_\xi := \Cap_{\delta,\gamma_\theta}(a)(a^\theta_\xi)$ and $z_\xi := \Cap_{\delta,\gamma_\theta}(a)(z^\theta_\xi)$. The construction and the subsequent verifications are the same as in the case $\xi = 0$, so we skip them. The only difference now is that, for each $\tau \in \mathrm{nacc}(\theta+1)$, we set $\varrho^\tau_\xi := (\sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}_{\gamma_\tau}(b_\eta \restriction \gamma_\tau))) + \omega + 1$.

Thereby, we get a condition a_ξ which is a legitimate move for **I** and, in response, **II** plays a condition b_ξ in D extending a_ξ and satisfying $b_\xi \leq^0_\delta s_\xi$. Once again, $a_\xi \restriction 1 = p_\xi$, $B_\xi \subseteq B_{a_\xi} \subseteq B_{b_\xi}$ and for all $\gamma \in B_\xi$,

$$(\dagger\dagger) \qquad (\sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}_\gamma(b_\eta \restriction \gamma))) + 1 < \mathrm{dom}(\mathsf{tp}_\gamma(a_\xi \restriction \gamma)).$$

Also, for all $i \in [\mathrm{dom}(\mathsf{tp}_\gamma(a \restriction \gamma)), \sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}_\gamma(b_\eta \restriction \gamma))]$,

$$(\dagger\dagger\dagger) \qquad \mathsf{tp}_\gamma(a_\xi \restriction \gamma)(i) \leq \mathrm{mtp}_\gamma(a \restriction \gamma).^{18}$$

At the end we obtain a sequence $\langle (a_\xi, b_\xi) \mid \xi < \chi\rangle$ which is a play in the game $\Game_{\mathbb{P}_\delta}(a, \vec{s}, D)$. By construction, for each $\xi < \chi$, $a_\xi \restriction 1 = p_\xi$, so that $\langle b_\xi \restriction 1 \mid \xi < \chi\rangle$ is diagonalizable with respect to $\vec{r}$. Let $p' \leq^0_1 \pi_{\delta,1}(a)$ be a witness for this latter fact.

Our next task is to show that $\langle b_\xi \mid \xi < \chi\rangle$ is diagonalizable and that the corresponding witness b fulfills the requirements of the lemma.

Claim 3.11.6. *The tuple $(a, \vec{r}, p', g, B_\chi)$ meets the requirements of Lemma 3.10, where $g : W_n(\pi_{\delta,1}(a)) \to \mathbb{P}_\delta \downarrow a$ is defined via $g(r_\xi) := b_\xi$.*

Proof. Let us go over the clauses of Lemma 3.10: Clause (0) holds by the construction of $\langle B_{b_\xi} \mid \xi < \chi\rangle$, Clause (1) is obvious and Clause (2) follows from the discussion of the previous paragraph. So, let us address the rest.

For each $\gamma \in B_\chi$, denote $\iota_\gamma := \min\{\xi < \chi \mid \gamma \in B_{b_\xi}\}$.

(3): Let $\gamma \in B_\chi$ and $\xi < \chi$:

- If $\xi < \iota_\gamma$ then $\gamma \notin B_{b_\xi}$ and so $b_\xi \restriction \gamma = \lceil b_\xi \restriction \beta\rceil^{\mathbb{P}_\gamma}$, where $\gamma = \beta + 1$. Thus, Lemma 3.6 and Definition 2.23(4) yield $\mathrm{dom}(\mathsf{tp}_\gamma(b_\xi \restriction \gamma)) = 0$.
- If $\xi = \iota_\gamma$, then $\gamma \in B_{b_\xi}$ and so $b_\xi \restriction \gamma \neq \lceil b_\xi \restriction \beta\rceil^{\mathbb{P}_\gamma}$, where $\gamma = \beta + 1$. Again, Lemma 3.6 and Definition 2.23(4) yield $\mathrm{dom}(\mathsf{tp}_\gamma(b_\xi \restriction \gamma)) \geq 1$.
- If $\xi > \iota_\gamma$, then $\gamma \in B_{b_{\iota_\gamma}} \subseteq B_\xi$.[19] Combining $(\dagger\dagger)$ above with $b_\xi \leq_\delta a_\xi$ and Clause (2) of Definition 2.23 we get

$$(\sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}_\gamma(b_\eta \restriction \gamma))) + 1 < \mathrm{dom}(\mathsf{tp}_\gamma(b_\xi \restriction \gamma)).$$

(4): Let $\gamma \in B_\chi$, $\iota_\gamma < \xi < \chi$ and i be as in Clause (4) of Lemma 3.10. By definition, $\gamma \in B_{b_{\iota_\gamma}} \subseteq B_\xi$, hence $(\dagger\dagger\dagger)$ yields $\mathsf{tp}_\gamma(a_\xi \restriction \gamma)(i) \leq \mathrm{mtp}_\gamma(a \restriction \gamma)$.

[18] For details about the verification of $(\dagger\dagger\dagger)$, see Claim 3.11.2.

[19] Recall that $B_\xi = \bigcup_{\eta<\xi} B_{b_\eta}$.

Combining this with Definition 2.23(3) and $b_\xi \leq_\delta a_\xi$ we arrive at

$$\mathsf{tp}_\gamma(b_\xi \restriction \gamma)(i) \leq \mathsf{tp}_\gamma(a_\xi \restriction \gamma)(i) \leq \mathrm{mtp}_\gamma(a \restriction \gamma).$$

(5): Let $\gamma \in B$. For all ξ such that $\iota_\gamma \leq \xi < \chi$, then $\gamma \in B_{b_\xi}$. Since for all such ξ's, b_ξ is a condition in $D \subseteq (\mathring{\mathbb{P}}_\delta)_{l+n}$ then $\mathrm{mtp}_\gamma(b_\xi \restriction \gamma) = 0$ (see Definition 3.8). Thus, clearly, $\sup_{\iota_\gamma \leq \xi < \chi} \mathrm{mtp}_\gamma(b_\xi \restriction \gamma) < \omega$.

The above completes the verification of the claim. □

Combining Claim 3.11.6 with Lemma 3.10 we get a condition b witnessing Clauses (a)–(c) of the latter. Note that thanks to (a) and (c) we can appeal to Proposition 2.22 with respect to $(\Cap_{\delta,1}, \pi_{\delta,1})$, a, $\vec{s}$, $\langle b_\xi \mid \xi < \chi\rangle$, p' and b and conclude that b diagonalizes $\langle b_\xi \mid \xi < \chi\rangle$ with respect to $\vec{s}$. □

Corollary 3.12. *For every $\delta \in \mathrm{acc}(\mu^+ + 1)$, if $(\mathring{\mathbb{P}}_\delta)_n$ forms a dense subposet of $(\mathbb{P}_\delta)_n$ for every $n < \omega$, then $(\mathbb{P}_\delta, \ell_\delta)$ has property $\mathcal{D}$.*

Proof. By Lemmas 3.11 and 2.11. □

The next lemma will useful in the proof of Lemma 3.14.

Lemma 3.13. *Let $\delta \in [2, \mu^+]$. Then, for all $n < \omega$ and every directed set D of conditions in $(\mathring{\mathbb{P}}_\delta)_n$ of size $< \kappa_n$, there is $q \in (\mathring{P}_\delta)_n$ such that q is a lower bound of D with $B_q = \bigcup_{p\in D} B_p$.*

Proof. We argue by induction on $2 \leq \delta \leq \mu^+$. The base case $\delta = 2$ can be proved similarly to the successor case below. So, we assume by induction that the statement holds for all $2 \leq \gamma < \delta$ and prove it for δ.

Fix an arbitrary directed family $D \subseteq (\mathring{P}_\delta)_n$ of size $< \kappa_n$.

► Suppose $\delta = \gamma + 1$. Then $\bar{D} := \{\pi_{\delta,\gamma}(p) \mid p \in D\}$ is a directed subset of $(\mathring{P}_\gamma)_n$ of size $< \kappa_n$, so that the inductive assumption yields a lower bound $p' \in (\mathring{P}_\gamma)_n$ for $\bar{D}$ such that $B_{p'} := \bigcup_{p\in D} B_{\pi_{\delta,\gamma}(p)}$. Set $\hat{D} := \{\Cap_{\delta,\gamma}(p)(p') \mid p \in D\}$, and note that $|\hat{D}| \leq |D| < \kappa_n$. By Lemma 3.6, $(\Cap_{\delta,\gamma}, \pi_{\delta,\gamma})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_\gamma, \ell_\gamma)$. So, Definition 2.13(7) together with Remark 2.24 imply that $\hat{D}$ is a directed subset of $(\mathring{P}_\delta)_n^{\pi_{\delta,\gamma}}$.

Recalling that $(\mathring{P}_\delta)_n^{\pi_{\delta,\gamma}}$ is isomorphic to the κ_n-directed-closed poset $\mathring{\mathbb{A}}_n^\pi$ given by Building Block II, we pick a lower bound $q \in (\mathring{P}_\delta)_n^{\pi_{\delta,\gamma}}$ for $\hat{D}$ such that $\pi_{\delta,\gamma}(q) = p'$. It is clear that q is the desired lower bound.

► Suppose δ is limit. Let $C := \mathrm{cl}(\bigcup_{p\in D} B_p) \cup \{1, \delta\}$. We shall define a $\subseteq$-increasing sequence $\langle p_\gamma \mid \gamma \in C\rangle \in \prod_{\gamma\in C}(\mathring{P}_\gamma)_n$ such that, for all $\gamma \in C$, p_γ is a lower bound for $\{\pi_{\delta,\gamma}(p) \mid p \in D\}$ with $B_{p_\gamma} = \bigcup_{p\in D} B_{\pi_{\delta,\gamma}(p)}$. Note that for each $\gamma \in C$, Lemma 3.9 yields $\{\pi_{\delta,\gamma}(p) \mid p \in D\} \subseteq (\mathring{P}_\gamma)_n$. We define the sequence $\langle p_\gamma \mid \gamma \in C\rangle$ by recursion on $\gamma \in C$:

- For $\gamma = 1$, $\{\pi_{\delta,1}(p) \mid p \in D\}$ is a directed subset of $(\mathring{P}_1)_n$ of size $< \kappa_n$. By Building Block I, $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, and hence we may find a lower bound $p_1 \in (\mathring{P}_1)_n$ for the set under consideration.

- Suppose $\gamma > 1$ is a non-accumulation point of $C \cap \delta$. Set $\gamma := \beta + 1$ and $\alpha := \sup(C \cap \gamma)$. Clearly, $\alpha \leq \beta$, so that Lemma 3.5(5) yields
$$\pitchfork_{\beta,\alpha}(\pi_{\delta,\beta}(p))(p_\alpha) = p_\alpha * \emptyset_\beta,$$
for each $p \in D$. Set $q := p_\alpha * \emptyset_\beta$ and note that the induction hypothesis on p_α yields $B_q = \bigcup_{p\in D} B_{\pi_{\delta,\beta}(p)}$. Set
$$\bar{D} := \{\pitchfork_{\gamma,\beta}(\pi_{\delta,\gamma}(p))(q) \mid p \in D\}.$$
Let $p \in D$. Then $\pi_{\delta,\gamma}(p) \in (\mathring{P}_\gamma)_n$ and by Lemma 3.9, $q \in (\mathring{P}_\beta)_n$. Also, by Lemma 3.6, tp_γ is a type over $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$, hence Remark 2.24 yields $\pitchfork_{\gamma,\beta}(\pi_{\delta,\gamma}(p))(q) \in (\mathring{P}_\gamma)_n$. Altogether, $\bar{D} \subseteq (\mathring{P}_\gamma)_n$.
Since $\bar{D}$ is a directed subset of $(\mathring{\mathbb{P}}_\gamma)_n$ of size $< \kappa_n$, arguing as in the successor case above we find $p_\gamma \in (\mathring{P}_\gamma)_n$ a lower bound for $\bar{D}$ with $\pi_{\gamma,\beta}(p_\gamma) = q$. By Lemma 3.5(2), $B_r = B_{\pi_{\delta,\gamma}(p)} \cup B_q$ for each $r \in \bar{D}$. Moreover, since $\gamma \in C$,
$$B_r = B_{\pi_{\delta,\beta}(p)} \cup \{\gamma\} \cup B_q = B_q \cup \{\gamma\},$$
for each $r \in \bar{D}$.
On one hand, since $p_\gamma \leq_\gamma r$, $B_r \subseteq B_{p_\gamma}$ and so $B_q \cup \{\gamma\} \subseteq B_{p_\gamma}$. On the other hand, since $\pi_{\gamma,\beta}(p_\gamma) = q$, $B_{p_\gamma} \subseteq B_q \cup \{\gamma\}$. Altogether, $B_{p_\gamma} = B_q \cup \{\gamma\}$ and thus $B_{p_\gamma} = \bigcup_{p\in D} B_{\pi_{\delta,\gamma}(p)}$. Also p_γ is a lower bound for $\{\pi_{\delta,\gamma}(p) \mid p \in D\}$. Finally, $\pi_{\gamma,\alpha}(p_\gamma) = \pi_{\beta,\alpha}(q) = p_\alpha$, and so the sequence $\langle p_{\bar{\gamma}} \mid \bar{\gamma} \leq \gamma\rangle$ is $\subseteq$-increasing.
- Suppose $\gamma \in \mathrm{acc}(C)$. Define $p_\gamma := \bigcup_{\beta\in(C\cap\gamma)} p_\beta$. By regularity of μ, we have $|B_{p_\gamma}| < \mu$, so that $p_\gamma \in P_\gamma$. Also, by the induction hypothesis, $B_{p_\gamma} = \bigcup_{p\in D} B_{\pi_{\delta,\gamma}(p)}$.
For all $p \in D$ and all $\beta \in C \cap \gamma$, we have $\pi_{\delta,\beta}(p_\gamma) = p_\beta \leq_\beta \pi_{\delta,\beta}(p)$, hence p_γ is a bound for $\{\pi_{\delta,\gamma}(p) \mid p \in D\}$ in $(P_\gamma)_n$.
We claim that $p_\gamma \in (\mathring{P}_\gamma)_n$: Let $\alpha \in B_{p_\gamma}$ and $\beta \in C \cap \gamma$ be such that $\alpha \in B_{p_\beta}$. By the induction hypothesis $p_\beta \in (\mathring{P}_\beta)_n$, hence Lemma 3.9 yields $\pi_{\gamma,\alpha}(p_\gamma) = \pi_{\beta,\alpha}(p_\beta) \in (\mathring{P}_\alpha)_n$. Also, by similar reasons, $\pi_{\gamma,1}(p_\gamma) = \pi_{\beta,1}(p_\beta) \in (\mathring{P}_1)_n$. Altogether, $p_\gamma \in (\mathring{P}_\gamma)_n$ and clearly, $\langle p_{\bar{\gamma}} \mid \bar{\gamma} \leq \gamma\rangle$ is $\subseteq$-increasing.
- Suppose $\gamma = \delta$, but $\gamma \notin \mathrm{acc}(C)$. In this case, let $\bar{\gamma} := \sup(C \cap \gamma)$, and then set $p_\gamma := p_{\bar{\gamma}} * \emptyset_\gamma$. As the interval $(\bar{\gamma}, \gamma]$ is disjoint from $\bigcup_{p\in D} B_p$, for every $p \in D$,
$$p_\gamma = p_{\bar{\gamma}} * \emptyset_\gamma \leq_\gamma \pi_{\gamma,\bar{\gamma}}(p) * \emptyset_\gamma = p.$$
Also, by the induction hypothesis, $p_{\bar{\gamma}} \in (\mathring{P}_{\bar{\gamma}})_n$ and $B_{p_\gamma} = B_{p_{\bar{\gamma}}} = \bigcup_{p\in D} B_p$. Finally, Lemma 3.9 yields $p_\gamma \in (\mathring{P}_\gamma)_n$. Note also that with this choice $\langle p_{\bar{\gamma}} \mid \bar{\gamma} \leq \gamma\rangle$ is $\subseteq$-increasing.

Clearly, p_δ is a lower bound for D in $(\mathring{\mathbb{P}}_\delta)_n$ with the desired property. □

We are now ready to address Clause (iv) of Goal 3.2.

Lemma 3.14. *For all nonzero $\delta \leq \mu^+$, $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$ satisfies all the requirements to be a Σ-Prikry triple, with the possible exceptions of Clause* (7) *and the density requirement in Clause* (2)*. Additionally, $\emptyset_\delta$ is the greatest condition in $\mathbb{P}_\delta$, $\ell_\delta = \ell_1 \circ \pi_{\delta,1}$, and $\emptyset_\delta \Vdash_{\mathbb{P}_\delta} \check{\mu} = \kappa^+$.*

Under the extra hypothesis that for each $\delta \in \mathrm{acc}(\mu^+ + 1)$ and every $n < \omega$, $(\mathring{\mathbb{P}}_\delta)_n$ is a dense subposet of $(\mathbb{P}_\delta)_n$, we have that for all nonzero $\delta \leq \mu^+$, $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$ is Σ-Prikry triple having property $\mathcal{D}$.

Proof. We argue by induction on δ. The base case $\delta = 1$ follows from the fact that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I. The successor step $\delta = \delta' + 1$ follows from the fact that $\mathbb{P}_{\delta'+1}$ was obtained by invoking Building Block II.

Next, suppose that $\delta \in \mathrm{acc}(\mu^+ + 1)$ is such that the conclusion of the lemma holds below δ. In particular, the hypothesis of Lemma 3.6 are satisfied, so that, for all nonzero $\beta \leq \gamma \leq \delta$, $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$ is a forking projection from $(\mathbb{P}_\gamma, \ell_\gamma)$ to $(\mathbb{P}_\beta, \ell_\beta)$. We now go over the clauses of Definition 2.3:

(1) The first bullet of Definition 2.1 follows from the fact that $\ell_\delta = \ell_1 \circ \pi_{\delta,1}$. Next, let $p \in P_\delta$ be arbitrary. Denote $\bar{p} := \pi_{\delta,1}(p)$. Since $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, we may pick $p' \leq_1 \bar{p}$ with $\ell_1(p') = \ell_1(\bar{p})+1$. As $(\pitchfork_{\delta,1}, \pi_{\delta,1})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_1, \ell_1)$, Fact 2.19(2) implies that $\pitchfork_{\delta,1}(p)(p')$ is an element of $(P_\delta)^p_1$.

(2) Let $n < \omega$. By Lemma 3.13, the poset $(\mathring{\mathbb{P}}_\delta)_n$ is κ_n-directed closed. Moreover, under the extra assumption that $(\mathring{\mathbb{P}}_\delta)_n$ is a dense subposet of $(\mathbb{P}_\delta)_n$ we have that $(\mathring{\mathbb{P}}_\delta)_n$ witnesses the statement of Clause (2).

The next claim takes care of Clause (3)

Claim 3.14.1. *Suppose $p, p' \in P_\delta$ with $c_\delta(p) = c_\delta(p')$. Then, $(P_\delta)^p_0 \cap (P_\delta)^{p'}_0$ is nonempty.*

Proof. If $\delta < \mu^+$, then since $(\pitchfork_{\delta,1}, \pi_{\delta,1})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$ to $(\mathbb{P}_1, \ell_1, c_1)$, we get from Clause (8) of Definition 2.13 that $c_1(p \restriction 1) = c_1(p' \restriction 1)$, and then by Clause (3) of Definition 2.3, we may pick $r \in (P_1)^{p\restriction 1}_0 \cap (P_1)^{p'\restriction 1}_0$. In effect, Clause (8) of Definition 2.13 entails $\pitchfork_{\delta,1}(p)(r) = \pitchfork_{\delta,1}(p')(r)$. Finally, Fact 2.19(2) implies that $\pitchfork_{\delta,1}(p)(r)$ is in $(P_\delta)^p_0$ and that $\pitchfork_{\delta,1}(p')(r)$ is in $(P_\delta)^{p'}_0$. In particular, $(P_\delta)^p_0 \cap (P_\delta)^{p'}_0$ is nonempty.

From now on, assume $\delta = \mu^+$. In particular, for all nonzero $\beta < \gamma < \mu^+$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a Σ-Prikry triple admitting a forking projection to $(\mathbb{P}_\beta, \ell_\beta, c_\beta)$ as witnessed by $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$. To avoid trivialities, assume also that $|\{\mathbb{1}_{\mu^+}, p, p'\}| = 3$. For each $q \in \{p, p'\}$, let $C_q := \mathrm{cl}(B_q)$ and define a function $e_q : C_q \to H_\mu$ via

$$e_q(\gamma) := (\phi_\gamma[C_q \cap \gamma], c_\gamma(q \restriction \gamma)).$$

Write i for the common value of $c_{\mu^+}(p)$ and $c_{\mu^+}(p')$. It follows that, for every $\gamma \in C_p \cap C_{p'}$, $e_p(\gamma) = e^i(\gamma) = e_{p'}(\gamma)$, so that $\phi_\gamma[C_p \cap \gamma] = \phi_\gamma[C_{p'} \cap \gamma]$ and hence $C_p \cap \gamma = C_{p'} \cap \gamma$. Consequently, $R := C_p \cap C_{p'}$ is an initial segment of C_p and an initial segment of $C_{p'}$.

Let $\zeta := \max(C_p \cup C_{p'})$, so that $p = (p \restriction \zeta) * \emptyset_{\mu^+}$ and $p' = (p' \restriction \zeta) * \emptyset_{\mu^+}$. Set $\gamma_0 := \max(\{0\} \cup R)$. By the above analysis, $C_p \cap (\gamma_0, \zeta]$ and $C_{p'} \cap (\gamma_0, \zeta]$ are two disjoint closed sets.

If $\gamma_0 = \zeta$, then $e_p(\zeta) = e_{p'}(\zeta)$, so that $c_\zeta(p \restriction \zeta) = c_\zeta(p' \restriction \zeta)$, and hence $(P_\zeta)_0^{p\restriction\zeta} \cap (P_\zeta)_0^{p'\restriction\zeta}$ is nonempty. Pick r in that intersection. Then $r * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0^p \cap (P_{\mu^+})_0^{p'}$.

Next, suppose that $\gamma_0 < \zeta$. Consequently, there exists a finite increasing sequence $\langle \gamma_{j+1} \mid j \leq k\rangle$ of ordinals from $C_p \cup C_{p'}$ such that $\gamma_{k+1} = \zeta$ and, for all $j \leq \kappa$:

(i) if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_p$;
(ii) if $\gamma_{j+1} \notin C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_{p'}$.

We now define a sequence $\langle r_j \mid j \leq k+1\rangle$ in $\prod_{j=0}^{k+1} \Big((P_{\gamma_j})_0^{p\restriction\gamma_j} \cap (P_{\gamma_j})_0^{p'\restriction\gamma_j}\Big)$, as follows.

- For $j = 0$, if $\gamma_0 \in C_p \cap C_{p'}$, then $e_p(\gamma_0) = e_{p'}(\gamma_0)$, so that $c_{\gamma_0}(p \restriction \gamma_0) = c_{\gamma_0}(p' \restriction \gamma_0)$, and we may indeed pick $r_0 \in (P_{\gamma_0})_0^{p\restriction\gamma_0} \cap (P_{\gamma_0})_0^{p'\restriction\gamma_0}$. If $\gamma_0 \notin C_p \cap C_{p'}$, then $\gamma_0 = 0$, and we simply let $r_0 := \emptyset$.
- Suppose that $j < k+1$, where r_j has already been defined. Let $q := \Cap_{\gamma_{j+1},\gamma_j}(p \restriction \gamma_{j+1})(r_j)$ and $q' := \Cap_{\gamma_{j+1},\gamma_j}(p' \restriction \gamma_{j+1})(r_j)$. By Lemma 3.5(2), $B_q = (B_p \cap \gamma_{j+1}) \cup B_{r_j}$ and $B_{q'} = (B_{p'} \cap \gamma_{j+1}) \cup B_{r_j}$. In particular, if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (B_q \cup B_{q'}) \subseteq B_q$, so that $q' = r_j * \emptyset_{\gamma_{j+1}}$ and $q \leq_{\gamma_{j+1}} q'$ by Clauses (5) and (6) of Lemma 3.5, respectively. Likewise, if $\gamma_{j+1} \notin C_p$, then $q = r_j * \emptyset_{\gamma_{j+1}}$, so that $q' \leq_{\gamma_{j+1}} q$. Thus, $\{q, q'\} \cap (P_{\gamma_j})_0^{p\restriction\gamma_j} \cap (P_{\gamma_j})_0^{p'\restriction\gamma_j}$ is nonempty, and we may let r_{j+1} be an element of that set.

Evidently, $r_{k+1} * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0^p \cap (P_{\mu^+})_0^{p'}$. □

(4) Let $p \in P_\delta$, $n, m < \omega$ and $q \in (P_\delta^p)_{n+m}$ be arbitrary. Recalling that $(\Cap_{\delta,1}, \pi_{\delta,1})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_1, \ell_1)$, we infer from Clause (4) of Definition 2.13 that $\Cap_{\delta,1}(p)(m(p \restriction 1, q \restriction 1))$ is the greatest element of $\{r \leq_\delta^n p \mid q \leq_\delta^m r\}$.

(5) Recalling that $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry, and that $(\Cap_{\delta,1}, \pi_{\delta,1})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_1, \ell_1)$, we infer from Fact 2.19(1) that, for every $p \in P_\delta$, $|W(p)| = |W(p \restriction 1)| < \mu$.

(6) Let $p', p \in P_\delta$ with $p' \leq_\delta p$. Let $q \in W(p')$ be arbitrary. For all $\gamma < \delta$, the pair $(\Cap_{\delta,\gamma}, \pi_{\delta,\gamma})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_\gamma, \ell_\gamma)$, so that by the special case $m = 0$ of Clause (4) of Definition 2.13,

$$w(p, q) = \Cap_{\delta,\gamma}(p)(w(p \restriction \gamma, q \restriction \gamma)).$$

Now, for all $q' \leq_\delta q$, the induction hypothesis implies that, for all $\gamma < \delta$, $w(p \restriction \gamma, q' \restriction \gamma) \leq_\gamma w(p \restriction \gamma, q \restriction \gamma)$. Together with Clause (5) of Definition 2.13, it follows that, for all $\gamma < \delta$,

$$w(p, q') \restriction \gamma = w(p \restriction \gamma, q' \restriction \gamma) \leq_\gamma w(p \restriction \gamma, q \restriction \gamma) = w(p, q) \restriction \gamma.$$

So, by the definition of $\leq_\delta$, $w(p,q') \leq_\delta w(p,q)$, as desired.

(7) By our assumptions, $(\Cap_{\delta,1}, \pi_{\delta,1})$ is a forking projection from $(\mathbb{P}_\delta, \ell_\delta)$ to $(\mathbb{P}_1, \ell_1)$ and $(\mathbb{P}_1, \ell_1, c_1)$ is Σ-Prikry. Moreover, under the extra assumption that for each $n < \omega$, $(\mathring{\mathbb{P}}_\delta)_n$ is a dense subposet of $(\mathbb{P}_\delta)_n$, Corollary 3.12 yields property $\mathcal{D}$ for $(\mathbb{P}_\delta, \ell_\delta)$. It thus follows from Lemma 2.21 that $(\mathbb{P}_\delta, \ell_\delta)$ has the CPP.

To complete our proof we shall need the following claim.

Claim 3.14.2. *For each δ with $1 \leq \delta \leq \mu^+$, $\mathbb{1}_{\mathbb{P}_\delta} \Vdash_{\mathbb{P}_\delta} \check{\mu} = \kappa^+$.*

Proof. The case $\delta = 1$ is given by Building Block I. Towards a contradiction, suppose that $1 < \delta \leq \mu^+$ and that $\mathbb{1}_{\mathbb{P}_\delta} \nVdash_{\mathbb{P}_\delta} \check{\mu} = \kappa^+$. As $\mathbb{1}_{\mathbb{P}_1} \Vdash_{\mathbb{P}_1} \check{\mu} = \kappa^+$ and $\mathbb{P}_\delta$ projects to $\mathbb{P}_1$, this means that there exists $p \in P_\delta$ such that $p \Vdash_{\mathbb{P}_\delta} |\mu| \leq |\kappa|$. Since $\mathbb{P}_1$ is isomorphic to the poset $\mathbb{Q}$ of Building Block I, and since $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "κ is singular",[20] $\mathbb{1}_{\mathbb{P}_1} \Vdash_{\mathbb{P}_1}$ "κ is singular". As $\mathbb{P}_\delta$ projects to $\mathbb{P}_1$, in fact $p \Vdash_{\mathbb{P}_\delta} \mathrm{cf}(\mu) < \kappa$. Thus, Lemma 2.7(2) yields a condition $p' \leq_\delta p$ with $|W(p')| \geq \mu$, contradicting Clause (5) above. □

This completes the proof of Lemma 3.14. □

4. An application

In this section, we present the first application of our iteration scheme. We will be constructing a model of finite simultaneous reflection at a successor of a singular strong limit cardinal κ in the presence of $\neg\,\mathrm{SCH}_\kappa$.

Definition 4.1. For cardinals $\theta < \mu = \mathrm{cf}(\mu)$ and stationary subsets S, Γ of μ, $\mathrm{Refl}(<\theta, S, \Gamma)$ stands for the following assertion. For every collection $\mathcal{S}$ of stationary subsets of S, with $|\mathcal{S}| < \theta$ and $\sup(\{\mathrm{cf}(\alpha) \mid \alpha \in \bigcup \mathcal{S}\}) < \mu$, there exists $\gamma \in \Gamma \cap E^\mu_{>\omega}$ such that, for every $S \in \mathcal{S}$, $S \cap \gamma$ is stationary in γ.

We write $\mathrm{Refl}(<\theta, S)$ for $\mathrm{Refl}(<\theta, S, \mu)$.

A proof of the following folklore fact may be found in [PRS20, §4].

Fact 4.2. *If κ is a singular strong limit cardinal admitting a stationary subset $S \subseteq \kappa^+$ for which $\mathrm{Refl}(<\mathrm{cf}(\kappa)^+, S)$ holds, then $2^\kappa = \kappa^+$.*

In particular, if κ is a singular strong limit cardinal of countable cofinality for which SCH_κ fails, and $\mathrm{Refl}(<\theta, \kappa^+)$ holds, then $\theta \leq \omega$. We shall soon show that $\theta := \omega$ is indeed feasible.

The following general statement about simultaneous reflection will be useful in our verification later on.

Proposition 4.3. *Suppose that μ is non-Mahlo cardinal, and $\theta \leq \mathrm{cf}(\mu)$. For stationary subsets T, Γ, R of μ, $\mathrm{Refl}(<2, T, \Gamma) + \mathrm{Refl}(<\theta, \Gamma, R)$ entails $\mathrm{Refl}(<\theta, T \cup \Gamma, R)$.*

[20] This is the sole part of the whole proof to make use of the fact that the poset given by Building Block I forces κ to be singular.

Proof. Given a collection $\mathcal{S}$ of stationary subsets of $T \cup \Gamma$, with $|\mathcal{S}| < \theta$ and $\sup(\{\mathrm{cf}(\alpha) \mid \alpha \in \bigcup \mathcal{S}\}) < \mu$, we shall first attach to any set $S \in \mathcal{S}$, a stationary subset S' of Γ, as follows.

▶ If $S \cap \Gamma$ is stationary, then let $S' := S \cap \Gamma$.

▶ If $S \cap \Gamma$ is nonstationary, then for every (sufficiently thin) club $C \subseteq \mu$, $S \cap C$ is a stationary subset of T, and so by $\mathrm{Refl}(<2, T, \Gamma)$, there exists $\gamma \in \Gamma \cap E^{\mu}_{>\omega}$ such that $(S \cap C) \cap \gamma$ is stationary in γ, and in particular, $\gamma \in C$. So, the set $\{\gamma \in \Gamma \mid S \cap \gamma \text{ is stationary}\}$ is stationary, and, as μ is non-Mahlo, we may pick S' which is a stationary subset of it and all of its points consists of the same cofinality.

Next, as $|\mathcal{S}| < \mathrm{cf}(\mu)$, we have $\sup(\{\mathrm{cf}(\gamma) \mid \gamma \in S', S \in \mathcal{S}\}) < \mu$, and so, from $\mathrm{Refl}(<\theta, \Gamma, R)$, we find some $\delta \in R$ such that $S' \cap \delta$ is stationary for all $S \in \mathcal{S}$.

Claim 4.3.1. *Let $S \in \mathcal{S}$. Then $S \cap \delta$ is stationary in δ.*

Proof. If $S' = S$, then $S \cap \delta = S' \cap \delta$ is stationary in δ, and we are done. Next, assume $S' \neq S$, and let c be an arbitrary club in δ. As $S' \cap \delta$ is stationary in δ, we may pick $\gamma \in \mathrm{acc}(c) \cap S'$. As $\gamma \in S' \subseteq E^{\mu}_{>\omega}$, $c \cap \gamma$ is a club in γ, and as $\gamma \in S'$, $S \cap \gamma$ is stationary, so $S \cap c \cap \gamma \neq \emptyset$. In particular, $S \cap c \neq \emptyset$. □

This completes the proof. □

4.1. **About Building Block II.** In this subsection, we describe Building Block II that we will be feeding to the iteration scheme of the preceding section. We were originally planning to use the functor given by [PRS20, §6], but unfortunately we found a gap in the proof of the mixing property [PRS20, Lemma 6.16]. To mitigate this gap, we shall relax Clause (4) of [PRS20, Definition 6.2] and prove that the outcome is a functor satisfying the weak mixing property (Lemma 4.16 below). Most of the results of [PRS20, §6] remain valid, as will be detailed later. Therefore, reading of this subsection does assume that the reader is comfortable with [PRS20, §6]. The upshot of this subsection is encapsulated by Corollary 4.18.

We commence by describing our setup for this subsection.

Setup 4. Suppose that we are given a Σ-Prikry notion of forcing $(\mathbb{P}, \ell, c)$ having property $\mathcal{D}$. Denote $\mathbb{P} = (P, \leq)$ and $\Sigma = \langle \kappa_n \mid n < \omega \rangle$. Also, define κ and μ as in Definition 2.3, and assume that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular" and that $\mu^{<\mu} = \mu$. Recall that for each $n < \omega$, we denote by $\mathring{\mathbb{P}}_n$ a dense κ_n-directed-closed subposet of $\mathbb{P}_n$. Our universe of sets is denoted by V, and we assume that, for all $n < \omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}(1, E^{\mu}_{\omega}, E^{\mu}_{<\kappa_n})$. Write $\Gamma := \{\gamma < \mu \mid \omega < \mathrm{cf}^V(\gamma) < \kappa\}$. We also assume that we are given a condition $r^\star$ forcing that $\dot{T}$ is a $\mathbb{P}$-name for some subset T of $(E^{\mu}_{\omega})^V$ such that, for all $\gamma \in \Gamma$, $T \cap \gamma$ is nonstationary in γ.

For each $n < \omega$, denote $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in E^{\mu}_{\omega} \times P_n \;\&\; p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}$.

Lemma 4.4. *For every $q \leq r^\star$, $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary".*

Proof. The proof is almost the same as that of [PRS20, Lemma 6.1], so we settle here for a sketch.

Suppose not, set $n := \ell(r^\star)$ and pick $p \leq^0 r^\star$ that $\mathbb{P}_n$-forces $\dot{T}_n$ is stationary. As $V^{\mathbb{P}_n} \models \mathrm{Refl}(1, E^\mu_\omega, E^\mu_{<\kappa_n})$, we may fix $p' \leq^0 p$ and $\gamma \in E^\mu_{<\kappa_n}$ of uncountable cofinality such that $p' \Vdash_{\mathbb{P}_n}$ "$\dot{T}_n \cap \gamma$ is stationary". As $\mathbb{P}_n$ has a κ_n-directed-closed dense subset, $\gamma \in \Gamma$, and there exists a ground model stationary subset B of γ such that

$$r \Vdash_{\mathbb{P}_n} \text{"}\dot{T}_n \cap \gamma \text{ contains the stationary set } \check{B}\text{"}.$$

By definition of the name $\dot{T}_n$, $r \Vdash_{\mathbb{P}} \check{B} \subseteq \dot{T} \cap \gamma$. Finally, as $\mathrm{otp}(B) < \kappa$, we infer from Lemma 2.7(1) that B remains stationary in any forcing extension by $\mathbb{P}$. So, $r \Vdash_{\mathbb{P}}$ "$\dot{T} \cap \gamma$ is stationary", contradicting the fact that $r \leq p' \leq p \leq r^\star$ and $\gamma \in \Gamma$. □

Set $I := \omega \setminus \ell(r^\star)$. By the preceding lemma, for each $n < \omega$, we may pick a $\mathbb{P}_n$-name $\dot{C}_n$ for a club subset of μ such that, for all $q \leq r^\star$ with $n := \ell(q)$ in I, $q \Vdash_{\mathbb{P}_n} \dot{T}_n \cap \dot{C}_n = \emptyset$. Let R be the binary relation

$$R := \{(\alpha, q) \in \mu \times P \mid q \leq r^\star \;\&\; \forall r \leq q[\ell(r) \in I \to r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)}]\},$$

and define $\dot{T}^+ := \{(\check{\alpha}, p) \mid (\alpha, p) \in (E^\mu_\omega)^V \times P \;\&\; p \Vdash_{\mathbb{P}_{\ell(p)}} \check{\alpha} \notin \dot{C}_{\ell(p)}\}$.

It is immediate that for all $q \leq r^\star$ with $n := \ell(q)$ in I, $q \Vdash_{\mathbb{P}} \dot{T} \subseteq \dot{T}^+$. Note that, for all $(\alpha, q) \in R$, $q \Vdash_{\mathbb{P}} \check{\alpha} \notin \dot{T}^+$. Also, if $(\alpha, q) \in R$ and $q' \leq q$ then $(\alpha, q') \in R$, as well.

In this section we will aim to kill the stationarity of the bigger set $\dot{T}^+$ in place of T. The whole point for this is that $\dot{T}^+$ has the following additional property: $q \leq r^\star$ with $n := \ell(q)$ in I, $q \Vdash_{\mathbb{P}_n} \dot{T}^+_n = \check{\mu} \setminus \dot{C}_n$. This will be crucial in our verification of density of the rings at limit stages (see Lemma 4.24 and, most concretely, the argument on page 52).

Definition 4.5 (relaxed form of [PRS20, Definition 6.2])**.** Suppose $p \in P$. A *labeled p-tree* is a function $S : W(p) \to [\mu]^{<\mu}$ such that for all $q \in W(p)$:

(1) $S(q)$ is a closed bounded subset of μ;
(2) $S(q') \supseteq S(q)$ whenever $q' \leq q$;
(3) $q \Vdash_{\mathbb{P}} S(q) \cap \dot{T}^+ = \emptyset$;
(4) there is a natural number m such that for any pair $q' \leq q$ of elements of $W(p)$, if $S(q') \neq \emptyset$ and $\ell(q) \geq \ell(p) + m$, then $(\max(S(q')), q) \in R$. The least such m is denoted by $m(S)$.

Remark 4.6. By Clause (4) and the Definition of R, for any pair $q' \leq q$ of elements of $W(p)$, if $\ell(q) \geq \ell(p) + m(S)$ and q is incompatible with $r^\star$, then $S(q') = \emptyset$.

Definition 4.7 ([PRS20, Definition 6.3])**.** For $p \in P$, we say that $\vec{S} = \langle S_i \mid i \leq \alpha \rangle$ is a *p-strategy* iff all of the following hold:

(1) $\alpha < \mu$;
(2) S_i is a labeled p-tree for all $i \leq \alpha$;
(3) for every $i < \alpha$ and $q \in W(p)$, $S_i(q) \sqsubseteq S_{i+1}(q)$;
(4) for every $i < \alpha$ and a pair $q' \leq q$ in $W(p)$, $(S_{i+1}(q) \setminus S_i(q)) \sqsubseteq (S_{i+1}(q') \setminus S_i(q'))$;
(5) for every limit $i \leq \alpha$ and $q \in W(p)$, $S_i(q)$ is the ordinal closure of $\bigcup_{j<i} S_j(q)$. In particular, $S_0(q) = \emptyset$ for all $q \in W(p)$.

Now, we are ready to describe our functor.

Definition 4.8 ([PRS20, Definition 6.4])**.** Let $\mathbb{A}(\mathbb{P}, \dot{T})$ be the notion of forcing $\mathbb{A} := (A, \trianglelefteq)$, where:

(1) $(p, \vec{S}) \in A$ iff $p \in P$, and $\vec{S}$ is either the empty sequence, or a p-strategy;
(2) $(p', \vec{S}') \trianglelefteq (p, \vec{S})$ iff:
  (a) $p' \leq p$;
  (b) $\mathrm{dom}(\vec{S}') \geq \mathrm{dom}(\vec{S})$;
  (c) $S'_i(q) = S_i(w(p,q))$ for all $i \in \mathrm{dom}(\vec{S})$ and $q \in W(p')$.

For all $p \in P$, denote $\lceil p \rceil^{\mathbb{A}} := (p, \emptyset)$.

Definition 4.9 ([PRS20, Definitions 6.10 and 6.11])**.**

- Define $c_{\mathbb{A}} : A \to H_\mu$ by letting, for all $(p, \vec{S}) \in A$,

$$c_{\mathbb{A}}(p, \vec{S}) := (c(p), \{(i, c(q), S_i(q)) \mid i \in \mathrm{dom}(\vec{S}), q \in W(p)\}).$$

- Define $\pi : A \to P$ by stipulating $\pi(p, \vec{S}) := p$ and $\ell_{\mathbb{A}} := \ell \circ \pi$.
- Given $a = (p, \vec{S})$ in A, define $\Cap(a) : \mathbb{P} \downarrow p \to A$ by letting for each $p' \leq p$, $\Cap(a)(p') := (p', \vec{S}')$, where $\vec{S}'$ is the sequence $\langle S'_i : W(p') \to [\mu]^{<\mu} \mid i < \mathrm{dom}(\vec{S})\rangle$ satisfying:

(*) $\quad S'_i(q) := S_i(w(p,q))$ for all $i \in \mathrm{dom}(\vec{S}')$ and $q \in W(p')$.

Even after relaxing Clause (4) of [PRS20, Definition 6.2] to that of Definition 4.5, the following remains valid, with essentially the same proofs.

Fact 4.10 ([PRS20, Corollary 4.13, Lemma 6.6, Theorem 6.8])**.**

(1) $\mathbb{1} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$*;*
(2) For every $\nu \geq \mu$*, if* $\mathbb{P}$ *is a subset of* H_ν*, then so is* $\mathbb{A}$*;*
(3) $\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}}$ "$\dot{T}^+$ *is nonstationary*"*.*[21]

Remark 4.11. Note that Fact 4.10(3) together with $r^\star \Vdash_{\mathbb{P}} \dot{T} \subseteq \dot{T}^+$ (see page 36) imply that $\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}}$ "$\dot{T}$ is nonstationary".

Lemma 4.12. $(\Cap, \pi)$ *is a forking projection from* $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ *to* $(\mathbb{P}, \ell, c)$*.*

[21]Here, Claim 4.15.1 below plays the role of [PRS20, Lemma 6.7]. Also, note that this is trivial when $\dot{T}^+$ is a $\mathbb{P}$-name for a nonstationary subset of μ in V.

Proof. The proof of [PRS20, Lemma 6.13] goes through, so we only focus on Clause (2) of Definition 2.13. Let $a \in A$ and $p' \leq \pi(a)$; we shall show that $\Cap(a)(p') \in A$ and $\Cap(a)(p') \trianglelefteq a$.

Write a as $(p, \vec{S})$. If $\vec{S} = \emptyset$, then $\Cap(a)(p') = \lceil p' \rceil^{\mathbb{A}}$, and we are done.

Next, suppose that $\mathrm{dom}(\vec{S}) = \alpha + 1$. Let $(p', \vec{S}') := \Cap(a)(p')$. Let $i \leq \alpha$ and we shall verify that S'_i is a p'-labeled tree with $m(S'_i) \leq m(S_i)$. We go over the clauses of Definition 4.5. To this end, let $q' \leq q$ be arbitrary pair of elements of $W(p')$.

(2) By Definition 2.3(6), we have $w(p, q') \leq w(p, q)$, so that $S'_i(q') = S_i(w(p, q')) \supseteq S_i(w(p, q)) = S'_i(q)$.

(3) As $q \leq w(p, q)$, $w(p, q) \Vdash_{\mathbb{P}} S_i(w(p, q)) \cap \dot{T} = \emptyset$, so that, since $S'_i(q) = S_i(w(p, q))$, we clearly have $q \Vdash_{\mathbb{P}} S'_i(q) \cap \dot{T} = \emptyset$.

(4) To avoid trivialities, Suppose that $S'_i(q') \neq \emptyset$ and $\ell(q) \geq m(S_i)$. Write $\gamma := \max(S'_i(q'))$. As $\ell(w(p, q)) = \ell(q) \geq m(S_i)$ and $\gamma = \max(S_i(w(p, q')))$, we infer that $(\gamma, w(p, q)) \in R$. In addition, $q \leq w(p, q)$, so by the definition of R it follows that $(\gamma, q) \in R$. Recalling that $\max(S'_i(q)) = \gamma$, we are done.[22]

To prove that $(p', \vec{S}')$ is a condition in A it now remains to argue that $\vec{S}'$ fulfills the requirements described in Clauses (3) and (5) of Definition 4.7 but this already follows from the definition of $\vec{S}'$ and the fact that $\vec{S}$ is a p-strategy. Finally $\Cap(a)(p') = (p', \vec{S}') \trianglelefteq (p, \vec{S}) = a$ by the very choice of p' and by Definition 4.9. □

We now introduce a type tp over $(\Cap, \pi)$ witnessing the weak mixing property.

Definition 4.13. Define a map $\mathsf{tp} : A \to {}^{<\mu}\omega$, as follows.

Given $a = (p, \vec{S})$ in A, write $\vec{S}$ as $\langle S_i \mid i < \beta \rangle$, and then let

$$\mathsf{tp}(a) := \langle m(S_i) \mid i < \beta \rangle.$$

We shall soon verify that tp is a type, but will use the mtp notation of Definition 2.23 from the outset. In particular, we will have $\mathring{\mathbb{A}} = (\mathring{A}, \trianglelefteq)$, with $\mathring{A} := \{a \in A \mid \pi(a) \in \mathring{P}_{\ell(\pi(a))} \ \&\ \mathrm{mtp}(a) = 0\}$. Note that the supercollection $\{a \in A \mid \mathrm{mtp}(a) = 0\}$ coincides with the set A from [PRS20, Definition 6.4]. In particular, the proof of [PRS20, Lemma 6.15] goes through, yielding the following fact.

Fact 4.14. *For all $n < \omega$, $\mathring{\mathbb{A}}^{\pi}_n$ is μ-directed-closed.* □

Lemma 4.15. *The map tp is a type over $(\Cap, \pi)$.*

Proof. We go over the clauses of Definition 2.23:

(1) This follows from the mere definition of tp.

[22] Following the terminology of Definition 4.5(4) note that here we have showed that $m(S'_i) \leq m(S_i)$. This will become important soon, whenever we introduce the type map associate to Sharon's functor (see Lemma 4.15).

(2) Write $b = (p', \vec{S}')$ and $a = (p, \vec{S})$. By Definitions 4.8 and 4.13, $\mathrm{dom}(\mathtt{tp}(b)) = \mathrm{dom}(\vec{S}') \geq \mathrm{dom}(\vec{S}) = \mathrm{dom}(\mathtt{tp}(a))$. Fix $i \in \mathrm{dom}(\mathtt{tp}(a))$ and let us show that $\mathtt{tp}(b)(i) \leq \mathtt{tp}(a)(i)$, i.e., that $m(S'_i) \leq m(S_i)$.

Let $q' \leq q$ be a pair of elements in $W(p')$ with $S'_i(q') \neq \emptyset$ and $\ell(q) \geq \ell(p') + m(S_i)$. By Definition 4.8(2c), $S'_i(q') = S_i(w(p,q'))$, hence it follows that $w(p,q') \leq w(p,q)$ is a pair of elements in $W(p)$ with $S_i(w(p,q')) \neq \emptyset$. Set $\gamma := \max(S_i(w(p,q')))$. By Definition 4.5(4), $(\gamma, w(p,q)) \in R$ hence the definition of R yields $(\gamma, q) \in R$. Noting that $\gamma = \max(S'_i(q'))$ it finally follows that $m(S'_i) \leq m(S_i)$.

(3) This follows from Definition 4.9(*).

(4) Let $a \in A$. If $a = \lceil \pi(a) \rceil^{\mathbb{A}}$ then $a = (\pi(a), \emptyset)$, and so $\mathtt{tp}(\lceil \pi(a) \rceil^{\mathbb{A}})$ is the empty sequence. Conversely, if $\mathtt{tp}(a)$ is the empty sequence then Definition 4.13 implies that a takes the form $(\pi(a), \emptyset)$, hence $a = \lceil \pi(a) \rceil^{\mathbb{A}}$.

(5) Write a as $(p, \langle S_i \mid i < \mathrm{dom}(\mathtt{tp}(a)) \rangle)$ and let $\alpha \in \mu \setminus \mathrm{dom}(\mathtt{tp}(a))$. There are two cases to consider:

▶ If $\mathrm{dom}(\mathtt{tp}(a)) = 0$, then let $a^\frown\alpha := (p, \langle T_i \mid i \leq \alpha \rangle)$, where $T_i : W(p) \to \{\emptyset\}$ is constant for every $i \leq \alpha$.

▶ Otherwise, say $\mathrm{dom}(\mathtt{tp}(a)) = \beta + 1$, let $a^\frown\alpha := (p, \langle T_i \mid i \leq \alpha \rangle)$, where $T_i := S_{\min\{i,\beta\}}$ for every $i \leq \alpha$.

It is routine to check that $a^\frown\alpha$ is as desired.

(6) Write $b = (p', \vec{S}')$ and $a = (p, \vec{S})$ and set $\gamma := \mathrm{dom}(\mathtt{tp}(b))$. If $\gamma = 0$ then $b^\frown\alpha \trianglelefteq a^\frown\alpha$ follows simply from $p' \leq p$. Otherwise, γ takes the form $\beta + 1$ and the above clause yields $b^\frown\alpha = (p', \vec{T}')$, where $\vec{T}' := \langle T'_i \mid i \leq \alpha \rangle$ and $T'_i := S'_{\min\{i,\beta\}}$. Similarly, $a^\frown\alpha = (p, \vec{T})$, where $\vec{T} := \langle T_i \mid i \leq \alpha \rangle$ and $T_i := S_{\min\{i,\beta\}}$. Using that $b \trianglelefteq a$, Definition 4.8 yields $b^\frown\alpha \trianglelefteq a^\frown\alpha$, as wanted.

(7) Let $a = (p, \vec{S}) \in A$. To avoid trivialities, let us assume that $\vec{S} \neq \emptyset$.

▶ Suppose p is incompatible with $r^\star$. Then, by Remark 4.6, for all $i < \mathrm{dom}(\mathtt{tp}(a))$ and all $q \in W(p)$, $S_i(q) = \emptyset$. Therefore, $\mathrm{mtp}(a) = 0$. Using Definition 2.3(2) find $p' \leq^0 p \in \mathring{P}$ and set $b := \pitchfork(a)(p')$. Combining Clauses (2) and (3) above with the fact that $\mathrm{mtp}(a) = 0$ it easily follows that $\mathrm{mtp}(b) = 0$. Also, $\pi(b) = p' \in \mathring{P}_{\ell(p)}$. Thus, $b \in \mathring{A}_{\ell(p)} \downarrow a$, as wanted.

▶ Suppose $p \leq r^\star$. The following claim will give us the desired condition.

Claim 4.15.1. *Let $\epsilon < \mu$. There exist $\alpha > \epsilon$ and $q \leq^0 \pi(a)$ such that $(\alpha, q) \in R$. Furthermore, there exist $\alpha > \epsilon$ and $b = (q, \vec{T}) \trianglelefteq^0 a$ such that $b \in \mathring{\mathbb{A}}$, $\mathrm{dom}(\vec{T}) = \alpha + 1$, and for all $r \in W(q)$, $\max(T_\alpha(r)) = \alpha$.*

Proof. Since $(\mathbb{P}, \ell, c)$ is Σ-Prikry, we infer from Definition 2.3(5) that $|W(p)| < \mu$. Thus, by possibly extending ϵ, we may assume that $S_i(q) \subseteq \epsilon$, for all $q \in W(p)$ and $i \in \mathrm{dom}(\mathtt{tp}(a))$. By Clause (5), we may also assume that $\mathrm{dom}(\mathtt{tp}(a))$ is a successor ordinal, say, it is $\beta + 1$.

As $p \leq r^\star$, by the very same proof of [PRS20, Claim 5.6.2(1)] and using Clause (2) of Definition 2.3, we may fix $(\alpha, q) \in R$ with $\alpha > \beta + \epsilon$, $q \leq^0 p$ and $q \in \mathring{P}_{\ell(p)}$. Define $\vec{T} = \langle T_i : W(q) \to [\mu]^{<\mu} \mid i \leq \alpha \rangle$ by letting for all

$r \in W(q)$ and $i \in \mathrm{dom}(\vec{T})$:

$$T_i(r) := \begin{cases} S_i(w(p,r)), & \text{if } i \leq \beta; \\ S_\beta(w(p,r)) \cup \{\alpha\}, & \text{otherwise.} \end{cases}$$

It is easy to see that T_i is a labeled q-tree for each $i \leq \alpha$. By Definitions 4.7, 4.8 and 4.9, we also have that $b = (q, \vec{T})$ is a condition in $\mathbb{A}$ with $b \trianglelefteq^0 a$ and $\pi(b) = q \in \mathring{P}_{\ell(p)}$. As $(\alpha, q) \in R$, then $(\alpha, r) \in R$ for all $r \in W(q)$, hence $\mathrm{mtp}(b) = 0$. Therefore, b is a condition in $\mathring{\mathbb{A}}$ with the desired properties. □

This completes the proof. □

Lemma 4.16 (Weak Mixing Property)**.** *For all $a \in A$, $n < \omega$, $\vec{r}$, and $p' \leq^0 \pi(a)$, and for every function $g : W_n(\pi(a)) \to \mathbb{A} \downarrow a$, if there exists an ordinal ι such that all of the following hold:*

(1) $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of $W_n(\pi(a))$;
(2) $\langle \pi(g(r_\xi)) \mid \xi < \chi \rangle$ is diagonalizable with respect to $\vec{r}$, as witnessed by p';
(3) for every $\xi < \chi$:
- *if $\xi < \iota$, then $\mathrm{dom}(\mathsf{tp}(g(r_\xi)) = 0$;*
- *if $\xi = \iota$, then $\mathrm{dom}(\mathsf{tp}(g(r_\xi)) \geq 1$;*
- *if $\xi > \iota$, then $\mathrm{dom}(\mathsf{tp}(g(r_\xi)) > (\sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}(g(r_\eta))) + 1$;*

(4) for all $\xi \in (\iota, \chi)$ and $i \in [\mathrm{dom}(\mathsf{tp}(a)), \sup_{\eta<\xi} \mathrm{dom}(\mathsf{tp}(g(r_\eta)))]$,

$$\mathsf{tp}(g(r_\xi))(i) \leq \mathrm{mtp}(a),$$

(5) $\sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi)) < \omega$,

then there exists $b \trianglelefteq^0 a$ with $\pi(b) = p'$ and $\mathrm{mtp}(b) \leq n + \sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi))$, such that for all $q' \in W_n(p')$,

$$\Cap(b)(q') \trianglelefteq^0 g(w(\pi(a), q')).$$

Proof. Let $a := (p, \vec{S})$. For each $\xi < \chi$, set $(p_\xi, \vec{S}^\xi) := g(r_\xi)$.

Claim 4.16.1. *If $\iota \geq \chi$ then there is $b \in A$ as in the lemma.*

Proof. If $\iota \geq \chi$ then Clause (3) yields $\mathrm{dom}(\mathsf{tp}(g(r_\xi)) = 0$ for all $\xi < \chi$. Hence, Clause (4) of Definition 2.23 yields $g(r_\xi) = \lceil p_\xi \rceil^{\mathbb{A}}$ for all $\xi < \chi$. In particular also $a = \lceil p \rceil^{\mathbb{A}}$. Set $b := \lceil p' \rceil^{\mathbb{A}}$, where p' is given by Clause (2).

Clearly, $\pi(b) = p'$ and $b \trianglelefteq^0 a$. Let $q' \in W_n(p')$. By Clause (2) above, $q' \leq^0 p_\xi$, where ξ is such that $r_\xi = w(p, q')$. Finally, Definition 2.13(6) yields $\Cap(b)(q') = \lceil q' \rceil^{\mathbb{A}} \trianglelefteq^0 \lceil p_\xi \rceil^{\mathbb{A}} = g(r_\xi)$, as desired. □

Hereafter let us assume that $\iota < \chi$. For each $\xi \in [\iota, \chi)$, Clause (3) and Definition 4.13 together imply that $\mathrm{dom}(\vec{S}^\xi) = \alpha_\xi + 1$ for some $\alpha_\xi < \mu$. Moreover, Clause (3) yields $\sup_{\iota \leq \eta < \xi} \alpha_\eta < \alpha_\xi$ for all $\xi \in (\iota, \chi)$. Likewise, the same clause implies that $g(r_\xi) = \lceil p_\xi \rceil^{\mathbb{A}}$, hence $\vec{S}^\xi = \emptyset$, for all $\xi < \iota$.

Let $\langle s_\tau \mid \tau < \theta\rangle$ be a good enumeration $W_n(p')$. By Fact 2.19, $\theta < \mu$. For each $\tau < \theta$, set $r_{\xi_\tau} := w(p, s_\tau)$. By Clause (1) above, for each $\tau < \theta$,

$$s_\tau \leq^0 \pi(g(w(p, s_\tau))) = \pi(g(r_{\xi_\tau})) = p_{\xi_\tau}.$$

Set $\alpha' := \sup_{\iota\leq\xi<\chi} \alpha_\xi$ and $\alpha := \sup(\mathrm{dom}(\vec{S}))$.[23] By regularity of μ and Clause (3) above it follows that $\alpha < \alpha' < \mu$. Our goal is to define a sequence $\vec{T} = \langle T_i : W(p') \to [\mu]^{<\mu} \mid i \leq \alpha'\rangle$ for which $b := (p', \vec{T})$ is a condition satisfying the conclusion of the lemma.

As $\langle s_\tau \mid \tau < \theta\rangle$ is a good enumeration of the n^{th}-level of the p'-tree $W(p')$, Fact 2.6 entails that, for each $q \in W(p')$, there is a unique ordinal $\tau_q < \theta$, such that q is comparable with s_{τ_q}. It thus follows from Fact 2.6(3) that, for all $q \in W(p')$, $\ell(q) - \ell(p') \geq n$ iff $q \in W(s_{\tau_q})$. Moreover, for each $q \in W_{\geq n}(p')$, $q \leq s_{\tau_q} \leq^0 p_{\xi_{\tau_q}}$, hence $w(p_{\xi_{\tau_q}}, q)$ is well-defined.

Now, for all $i \leq \alpha'$ and $q \in W(p')$, let:

$$T_i(q) := \begin{cases} S^{\xi_{\tau_q}}_{\min\{i,\alpha_{\xi_{\tau_q}}\}}(w(p_{\xi_{\tau_q}}, q)), & \text{if } q \in W(s_{\tau_q}) \ \&\ \iota \leq \xi_{\tau_q};\\ S_{\min\{i,\alpha\}}(w(p, q)), & \text{if } q \notin W(s_{\tau_q}) \ \&\ \alpha > 0;\\ \emptyset, & \text{otherwise.}\end{cases}$$

Claim 4.16.2. *Let $i \leq \alpha'$. Then T_i is a labeled p'-tree.*

Proof. Fix $q \in W(p')$ and let us go over the Clauses of Definition 4.5. The verification of (1), (2) and (3) are similar to that of [PRS20, Claim 6.16.1], so we just provide details for the new Clause (4).

For each $i < \alpha'$, set

$$\xi(i) := \min\{\xi \in [\iota, \chi) \mid i \leq \alpha_\xi\}.$$

Subclaim 4.16.2.1. *If $i < \alpha'$, then*

$$m(T_i) \leq n + \max\{\mathrm{mtp}(a), \sup_{\iota\leq\eta<\xi(i)} \mathrm{mtp}(g(r_\eta)), \mathsf{tp}(g(r_{\xi(i)})(i)\}.$$

Proof. Let $q' \leq q$ be in $W(p')$ with $q \in W_k(p')$, where

$$k \geq n + \max\{\mathrm{mtp}(a), \sup_{\iota\leq\eta<\xi(i)} \mathrm{mtp}(g(r_\eta)), \mathsf{tp}(g(r_{\xi(i)})(i)\}.$$

Suppose that $T_i(q') \neq \emptyset$. Denote $\tau := \tau_{q'}$ and $\delta := \max(T_i(q'))$. Since $\ell(q) \geq \ell(p') + n$, note that $q, q' \in W(s_\tau)$. Also, $\iota \leq \xi_\tau$, as otherwise $T_i(q') = \emptyset$. Therefore, we fall into the first option of the casuistic getting

$$T_i(q') = S^{\xi_\tau}_{\min\{i,\alpha_{\xi_\tau}\}}(w(p_{\xi_\tau}, q')).$$

▶ Assume that $\xi_\tau < \xi(i)$. Then, $\alpha_{\xi_\tau} < i$ and so

$$T_i(q') = S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q')).$$

We have that $w(p_{\xi_\tau}, q') \leq w(p_{\xi_\tau}, q)$ is a pair in $W_{k-n}(p_{\xi_\tau})$ and that the set $S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q'))$ is non-empty. Also, $k - n \geq \mathrm{mtp}(g(r_{\xi_\tau})) = m(S^{\xi_\tau}_{\alpha_{\xi_\tau}})$. So, by Clause (4) for $S^{\xi_\tau}_{\alpha_{\xi_\tau}}$, we have that $(\delta, w(p_{\xi_\tau}, q)) \in R$, and thus $(\delta, q) \in R$.

[23] Note that a might be $\lceil p\rceil^{\mathbb{A}}$, so we are allowing $\alpha = 0$.

▶ Assume that $\xi(i) \leq \xi_\tau$. Then $i \leq \alpha_{\xi(i)} \leq \alpha_{\xi_\tau}$, and thus

$$T_i(q') = S_i^{\xi_\tau}(w(p_{\xi_\tau}, q')).$$

If $\mathrm{dom}(\mathsf{tp}(a)) \leq i \leq \sup_{\iota \leq \eta < \xi(i)} \alpha_\eta$, by Clause (4) above,

$$\mathsf{tp}(g(r_{\xi_\tau}))(i) \leq \mathrm{mtp}(a).$$

Otherwise, if $\sup_{\iota \leq \eta < \xi(i)} \alpha_\eta < i \leq \alpha_{\xi(i)}$, again by Clause (4) above

$$\mathsf{tp}(g(r_{\xi_\tau}))(i) \leq \max\{\mathrm{mtp}(a), \mathsf{tp}(g(r_{\xi(i)})(i)\}.$$

In either case, $w(p_{\xi_\tau}, q) \in W_{k-n}(p_{\xi_\tau})$ and $k - n \geq \mathsf{tp}(g(r_{\xi_\tau}))(i) = m(S_i^{\xi_\tau})$. So by Clause (4) of $S_i^{\xi_\tau}$ we get that $(\delta, w(p_{\xi_\tau}, q)) \in R$, hence $(\delta, q) \in R$. □

Subclaim 4.16.2.2. $m(T_{\alpha'}) \leq n + \sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi))$.

Proof. Let $q' \leq q$ be in $W(p')$ with $q \in W_k(p')$ and $k \geq n + \sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi))$, and suppose that $T_{\alpha'}(q') \neq \emptyset$. Denote $\tau := \tau_{q'}$ and $\delta := \max(T_{\alpha'}(q'))$.

Since $k \geq n$, $q, q' \in W(s_\tau)$. Also, $\iota \leq \xi_\tau$, as otherwise $T_{\alpha'}(q') = \emptyset$. Hence, $T_{\alpha'}(q') = S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q'))$. Then $w(p_{\xi_\tau}, q') \leq w(p_{\xi_\tau}, q)$ is a pair in $W_{k-n}(p_{\xi_\tau})$ with $k - n \geq \mathrm{mtp}(g(r_{\xi_\tau})) = m(S^{\xi_\tau}_{\alpha_{\xi_\tau}})$. So, by Definition 4.5(4) regarded with respect to $S^{\xi_\tau}_{\alpha_{\xi_\tau}}$, it follows that $(\delta, w(p_{\xi_\tau}, q)) \in R$. Thus, $(\delta, q) \in R$, as wanted. □

The combination of the above subclaims yield Clause (4) for T_i. □

Claim 4.16.3. *The sequence $\vec{T} = \langle T_i : W(p') \to [\mu]^{<\mu} \mid i \leq \alpha' \rangle$ is a p'-strategy.*

Proof. We need to go over the clauses of Definition 4.7. However, Clause (1) is trivial, Clause (2) is established in the preceding claim, and Clauses (3) and (5) follow from the corresponding features of $\vec{S}$ and the $\vec{S}^{r^\tau}$'s. Finally, Clause (4) can be proved similarly to [PRS20, Claim 6.16.2], noting that if $\alpha > 0$ then $\iota = 0$. □

Thus, we have established that $b := (p', \vec{T})$ is a legitimate condition in $\mathbb{A}$, such that $\mathrm{mtp}(b) \leq n + \sup_{\xi < \chi} \mathrm{mtp}(g(r_\xi))$.

The next series of claims take care of the rest of the lemma:

Claim 4.16.4. *Let $\tau < \theta$. For each $q \in W_n(s_\tau)$, $w(p', q) = w(s_\tau, q) = q$.*

Proof. The first equality can be proved exactly as in [PRS20, Claim 6.16.4]. For the second, notice that q and $w(s_\tau, q)$ are conditions in $W(s_\tau)$ with the same length. Hence, Fact 2.6(2) yields $q = w(s_\tau, q)$, as wanted. □

Claim 4.16.5. $\pi(b) = p'$ *and* $b \trianglelefteq^0 a$.

Proof. The proof of this can be found in [PRS20, Claim 6.16.3]. □

Claim 4.16.6. *For each $\tau < \theta$, $\Cap(b)(s_\tau) \trianglelefteq^0 g(r_{\xi_\tau})$.*[24]

[24]Recall that $\langle s_\tau \mid \tau < \theta \rangle$ was a good enumeration of $W_n(p')$.

Proof. Let $\tau < \theta$ and $\vec{T}^\tau$ be denote the s_τ-strategy such that $\pitchfork(b)(s_\tau) = (s_\tau, \vec{T}_\tau)$. By Corollary 4.12 we have that $\pi(\pitchfork(b)(s_\tau)) = s_\tau \leq^0 p_{\xi_\tau}$.

If $\xi_\tau < \iota$, then $\pitchfork(b)(s_\tau) \trianglelefteq^0 \lceil p_{\xi_\tau} \rceil^{\mathbb{A}} = g(r_{\xi_\tau})$, and we are done.

So, let us assume that $\iota \leq \xi_\tau$. Let $i \leq \alpha_{\xi_\tau}$ and $q \in W(s_\tau)$. By Definition 4.9(*), $T^\tau_i(q) = T_i(w(p', q))$ and by one of the preceding claims, $w(p', q) = w(s_\tau, q) = q$, hence $T^\tau_i(q) = T_i(q)$. Also $r_{\xi_{\tau_q}} = w(p, s_{\tau_q}) = w(p, s_\tau) = r_{\xi_\tau}$, where the second last equality follows from $q \in W(s_\tau)$. Therefore,

$$T^\tau_i(q) = S^{\xi_\tau}_{\min\{i,\alpha_{\xi_\tau}\}}(w(p_{\xi_\tau}, q)) = S^{\xi_\tau}_i(w(p_{\xi_\tau}, q)).$$

Altogether, $\pitchfork(b)(s_\tau) \trianglelefteq^0 g(r_{\xi_\tau})$, as wanted. □

The above claims yield the proof of the lemma. □

Combining Lemmas 4.12 and 4.16 we arrive at:

Corollary 4.17. *$(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell, c)$ having the weak mixing property.* □

Now we take advantage of the preceding corollary to establish that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is Σ-Prikry and that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$. On this respect, note that the latter statement follows combining Corollary 4.17, Lemma 2.27 and property $\mathcal{D}$ of $(\mathbb{P}, \ell)$ (Setup 4). For the former let us go over the clauses of Definition 2.3: Clauses (1),(3),(4),(5) and (6) follow from lemmas 4.5, 4.7, 4.8 and 4.9 of [PRS20], respectively. Clause (7) follows combining property $\mathcal{D}$ of $(\mathbb{P}, \ell)$ with Corollary 4.17 and Corollary 2.28. Also, by [PRS20, Corollary 4.13], $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$. Finally, note that Clause (2) follows from Lemma 2.29 together with Corollary 4.17 and Fact 4.14.

Altogether, we arrive at the main result of this section:

Corollary 4.18. *Suppose:*

- *(i) $(\mathbb{P}, \ell, c)$ is a Σ-Prikry notion of forcing such that the pair $(\mathbb{P}, \ell)$ has property $\mathcal{D}$;*
- *(ii) $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$;*
- *(iii) $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+};*
- *(iv) $r^\star \in P$ forces that $\dot{T}$ is a $\mathbb{P}$-name for some subset T of $(E^\mu_\omega)^V$ such that, for all $\gamma < \mu$ with $\omega < \mathrm{cf}^V(\gamma) < \kappa$, $T \cap \gamma$ is nonstationary in γ.*

Then, there exists a Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ such that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$ and for which the following are true:

- *(1) $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection $(\pitchfork, \pi)$ to $(\mathbb{P}, \ell, c)$ that has the weak mixing property;*
- *(2) for each $n < \omega$, $\mathring{\mathbb{A}}^\pi_n$ is μ-directed-closed;*
- *(3) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \kappa^+$;*
- *(4) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};*
- *(5) $\lceil r^\star \rceil^{\mathbb{A}}$ forces that $\dot{T}$ is nonstationary.*

Proof. Item (1) and the assertion that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is Σ-Prikry and that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$ follow from our previous arguments. Item (2) follows from Fact 4.14 and items (3),(4) and (5) already appeared in Fact 4.10.[25] □

4.2. **Connecting the dots.** For the rest of this section, we make the following assumptions:

- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;
- $\Gamma := \{\gamma < \mu \mid \omega < \mathrm{cf}^V(\gamma) < \kappa\}$.

Under these assumptions, [PRS20, Corollary 5.11] reads as follows:

Fact 4.19. *If $(\mathbb{P}, \ell, c)$ is a Σ-Prikry notion of forcing such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, then $V^{\mathbb{P}} \models \mathrm{Refl}(<\omega, \Gamma)$.*

We now want to appeal to the iteration scheme of the previous section. For this, we need to introduce our three building blocks of choice.

Building Block I. Let $\mathbb{Q}$ be the Extender Based Prikry Forcing (EBPF) for blowing up 2^κ to κ^{++}. By results in [Pov20, Ch.10, §2.5], this notion of forcing can be regarded as a Σ-Prikry triple $(\mathbb{Q}, \ell, c)$ for which $(\mathbb{Q}, \ell)$ has property $\mathcal{D}$, $\mathbb{Q}$ is a subset of H_{μ^+}, and $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \check{\mu} = \kappa^+$. Furthermore, for each $n < \omega$, $\mathbb{Q}_n$ is κ_n-directed-closed, so we set $\mathring{\mathbb{Q}}_n := \mathbb{Q}_n$. Finally, as κ is singular, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "κ is singular".

Building Block II. For every Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ having property $\mathcal{D}$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+} and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$, every $r^\star \in P$, and every $\mathbb{P}$-name $z \in H_{\mu^+}$, we obtain a corresponding Σ-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ as follows:

- ▶ If $r^\star \in P$ forces that z is a $\mathbb{P}$-name for a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, then we first let σ be a $\mathbb{P}$-*nice name* for a subset of $(E^\mu_\omega)^V$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma = z$. Setting $\dot{T} := \{(\check{\alpha}, p) \in \sigma \mid p \text{ is compatible with } r^\star\}$, we then get that $r^\star \Vdash_{\mathbb{P}} \dot{T} = z$. Furthermore, $\mathbb{1}_{\mathbb{P}}$ forces that $\dot{T}$ is a $\mathbb{P}$-name for some subset T of $(E^\mu_\omega)^V$ such that, for all $\gamma \in \Gamma$, $T \cap \gamma$ is nonstationary in γ. We then obtain $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ by appealing to Corollary 4.18 with the Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, the condition $\mathbb{1}_{\mathbb{P}}$ and the name $\dot{T}$. In effect, $\lceil \mathbb{1}_{\mathbb{P}} \rceil^{\mathbb{A}}$ forces that $\dot{T}$ is nonstationary, so that $\lceil r^\star \rceil^{\mathbb{A}}$ forces that z is nonstationary.
- ▶ Otherwise, we invoke Corollary 4.18 with the Σ-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, the condition $\mathbb{1}_{\mathbb{P}}$ and the name $\dot{T} := \emptyset$.

In either case, we get:

(a) $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ admits a forking projection $(\pitchfork, \pi)$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ that has the weak mixing property;

[25] See also Remark 4.11.

(b) for each $n<\omega$, $\mathring{\mathbb{A}}^{\pi}_n$ is κ_n-directed-closed;[26]
(c) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \kappa^+$;
(d) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};
(e) if $r^\star$ forces that z is a $\mathbb{P}$-name for a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, then $\lceil r^\star \rceil^{\mathbb{A}}$ forces that z is nonstationary.

Building Block III. As $2^\mu=\mu^+$, we fix a surjection $\psi:\mu^+\to H_{\mu^+}$ such that the preimage of any singleton is cofinal in μ^+.

We would like now to appeal to the iteration scheme of Section 3 with these building blocks. However, Lemma 3.14 partially bears on the extra assumption that for all $\delta\in\mathrm{acc}(\mu^+ + 1)$ and $n<\omega$, $(\mathring{\mathbb{P}}_\delta)_n$ is dense in $(\mathbb{P}_\delta)_n$. Our next task will be checking that the iteration defined using the previous building blocks has this feature. Once we are done we will prove Theorem 4.25, which yields the very first application of our iteration scheme.

Definition 4.20. For every nonzero $\beta<\mu$, as $(\mathbb{P}_{\beta+1},\ell_{\beta+1},c_{\beta+1})$ is obtained from $(\mathbb{P}_\beta,\ell_\beta,c_\beta)$ by invoking the above-mentioned Building Block II, we shall denote by $\langle \dot{C}^\beta_n \mid n<\omega\rangle$, and R_β the corresponding objects appearing before Definition 4.5 and involved in defining $\mathbb{P}_{\beta+1}$. In particular, $R_\beta\subseteq \mu\times P_\beta$. We shall denote by $(\dot{T}^+)^\beta$ the $\mathbb{P}_\beta$-name defined as in page 36. More explicitly,

$$(\dot{T}^+)^\beta := \{(\check{\alpha},p)\mid (\alpha,p)\in (E^\mu_\omega)^{V^{\mathbb{P}_\beta}}\times P_\beta \;\&\; (p\Vdash_{(\mathbb{P}_\beta)_{\ell(p)}} \check{\alpha}\in \dot{C}^\beta_{\ell(p)})\}.$$

Remark 4.21. Recall that the $\mathbb{P}_\beta$-name $(\dot{T}^+)^\beta$ have the following additional property: for all $q\in P_\beta$, setting $n:=\ell(q)$, $q\Vdash_{\mathbb{P}_n} (\dot{T}^+)^\beta_n=\check{\mu}\setminus \dot{C}^\beta_n$. This will be crucially used in the argument towards verifying Clause (4) for S_α. More precisely, when we show on page 52 that $(\epsilon_\omega,\lceil p\rceil^{\mathbb{P}_{\beta_0}})\in R$.

Definition 4.22. Let $\delta\in[2,\mu^+]$ and $a\in P_\delta$.

- For every nonzero $\beta<\delta$, if $\beta+1\in B_a$, then $(a\restriction\beta+1)=(a\restriction\beta)^\frown\langle\vec{S}\rangle$ for some nonempty sequence $\vec{S}$, so we denote this sequence by $\vec{S}^{a,\beta}=\langle S^{a,\beta}_i\mid i\leq\alpha^{a,\beta}\rangle$;
- For every nonzero $\beta<\delta$ such that $\beta+1\in B_a$, and every $q\in W(a\restriction\beta)$, we let $\sigma^{a,\beta}(q):=\max(\{0\}\cup S^{a,\beta}_{\alpha^{a,\beta}}(q))$;
- Let $\sigma^a:=\sup\{\sigma^{a,\beta}(q)\mid \beta+1\in B_a\setminus 2\;\&\; q\in W(a\restriction\beta)\}$.

Lemma 4.23. *Let $\delta\in[2,\mu^+]$, $a\in P_\delta$ and $\beta+1\in B_a\setminus 2$.*

(1) $\sigma^{a,\beta}:W(a\restriction\beta)\to\mu$ is order-reversing; In particular, for all $q\in W(a\restriction\beta)$, $\sigma^{a,\beta}(a\restriction\beta)\leq\sigma^{a,\beta}(q)\leq\sigma^a$;
(2) For all $r\leq_\beta a\restriction\beta$ and $s\in W(r)$, letting $b:=\Cap_{\beta+1,\beta}(a\restriction(\beta+1))(r)$,

$$\sigma^{b,\beta}(s)=\sigma^{a,\beta}(w(a\restriction\beta,s));$$

[26]Recall Footnote 9 on page 15.

(3) For all $b \leq_{\beta+1} a \restriction (\beta+1)$ *and* $q \in W(b \restriction \beta)$,

$$\sigma^{b,\beta}(q) \geq \sigma^{a,\beta}(w(a \restriction \beta, q)).$$

Proof. (1) By Definition 4.5(2).

(2) By Definition 4.9(*).

(3) By Definition 4.7(3) and Definition 4.9. □

Lemma 4.24. *For all* $\delta \in [2, \mu^+]$, $n < \omega$, *and* $\epsilon < \mu$,

$$D^{\epsilon}_{\delta,n} := \{b \in (\mathring{P}_\delta)_n \mid \forall \beta+1 \in B_b \setminus 2\,[\epsilon < \sigma^{b,\beta}(b \restriction \beta)]\}$$

is dense in $(\mathbb{P}_\delta)_n$.

In particular, for all $\delta \in [2, \mu^+]$ *and* $n < \omega$, $(\mathring{\mathbb{P}}_\delta)_n$ *is dense in* $(\mathbb{P}_\delta)_n$.

Proof. By induction on δ. Suppose that we are given $\delta \in [2, \mu^+]$ such that for all $\gamma \in [2, \delta)$, $n < \omega$ and $\epsilon < \mu$, $D^{\epsilon}_{\gamma,n}$ is dense in $(\mathbb{P}_\gamma)_n$.

Case 1: Suppose that $\delta = \beta + 1$ is a successor ordinal. Let $a \in P_\delta$ and $\epsilon < \mu$ be arbitrary. Denote $n := \ell_\delta(a)$. Appealing to Claim 4.15.1, find $a' \leq^0_\beta (a \restriction \beta)$ and $\alpha > \max\{\epsilon, \sigma^a\}$ with $(\alpha, a') \in R_\beta$.

If $\beta = 1$, then set $b' := a'$; otherwise, appeal to the inductive hypothesis to pick $b' \in D^{\epsilon}_{\beta,n}$ extending a'. In either case, $(\alpha, b') \in R_\beta$.

If $\beta + 1 \notin B_a$, then we are done by setting $b := b' * \emptyset_\delta$, so suppose that $\beta + 1 \in B_a$. In particular, $a = (a \restriction \beta)^\frown \langle \langle S^{a,\beta}_i \mid i \leq \alpha^{a,\beta} \rangle \rangle$. Now, let $b := b'^\frown \langle \langle S_i \mid i \leq \alpha^{a,\beta} + 1 \rangle \rangle$, where for each $q \in W(b')$, $S_i(q)$ is defined as follows:

$$S_i(q) := \begin{cases} S^{a,\beta}_i(w(a \restriction \beta, q)), & \text{if } i \leq \alpha^{a,\beta}; \\ S^{a,\beta}_i(w(a \restriction \beta, q)) \cup \{\alpha\}, & \text{otherwise.} \end{cases}$$

As $(\alpha, b \restriction \beta) = (\alpha, b') \in R_\beta$, we infer that $b \in P_\delta$ and $\mathrm{mtp}_{\beta+1}(b) = 0$. Thus, since $b' \in \mathring{P}_\beta$, it follows that $b \in \mathring{P}_\delta$. Finally, since $B_b = B_{b'} \cup \{\beta+1\}$, our choice of b' implies that b is an element of $D^{\epsilon}_{\delta,n}$ extending a.

Case 2: Suppose that $\mathrm{cf}(\delta) > \kappa$. Let $a \in P_\delta$ and $\epsilon < \mu$ be arbitrary. Denote $n := \ell_\delta(a)$. Then B_a is bounded in δ. Fix $\gamma < \delta$ such that $a = (a \restriction \gamma) * \emptyset_\delta$. By the inductive hypothesis, we find $a' \in D^{\epsilon}_{\gamma,n}$ extending $a \restriction \gamma$. Set $b := a' * \emptyset_\delta$, so that $B_b = B_{a'}$. Then $b \in D^{\epsilon}_{\delta,n}$ extends a, as desired.

Case 3: Suppose that $1 < \mathrm{cf}(\delta) \leq \kappa$. As κ is the limit of the strictly increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ (recall the opening of Subsection 4.2), we may let $m < \omega$ be the least such that $\mathrm{cf}(\delta) < \kappa_m$.

Claim 4.24.1. *For all* $n \geq m$ *and* $\epsilon < \mu$, $D^{\epsilon}_{\delta,n}$ *is dense in* $(\mathbb{P}_\delta)_n$.

Proof. Let $\epsilon < \mu$ and let $a \in P_\delta$ be such that $n := \ell_\delta(a)$ is $\geq m$. By the proof of Case 2, we may assume that B_a is unbounded in δ. Let $\langle \gamma_\tau \mid \tau < \mathrm{cf}(\delta) \rangle$ be the increasing enumeration of a small cofinal subset of $B_a \setminus 2$, and set $\gamma_{\mathrm{cf}(\delta)} := \delta$. We now construct a sequence of conditions $\langle b_\tau \mid \tau \leq \mathrm{cf}(\delta) \rangle \in \prod_{\tau \leq \mathrm{cf}(\delta)} D^{\epsilon}_{\gamma_\tau,n}$ in such a way that, for all $\eta < \tau \leq \mathrm{cf}(\delta)$, $b_\eta \leq^0_{\gamma_\eta} a \restriction \gamma_\eta, b_\tau \restriction \gamma_\eta$. The construction is by recursion on $\tau \leq \mathrm{cf}(\delta)$, as follows:

▶ For $\tau = 0$, use the induction hypothesis to find $b_0 \leq^0_{\gamma_0} a \restriction \gamma_0$ in $D^\epsilon_{\gamma_0,n}$.

▶ For every $\tau < \mathrm{cf}(\delta)$ such that b_τ has already been defined, use the induction hypothesis to find $b_{\tau+1} \leq^0_{\gamma_{\tau+1}} \Cap_{\gamma_{\tau+1},\gamma_\tau}(a \restriction \gamma_{\tau+1})(b_\tau)$ in $D^\epsilon_{\gamma_{\tau+1},n}$.

▶ For every $\tau \in \mathrm{acc}(\mathrm{cf}(\delta)+1)$ such that $\langle b_\eta \mid \eta < \tau\rangle$ has already been defined, we get from the induction hypothesis together with Clause (4) of Definition 2.23 that $\langle b_\eta * \emptyset_{\gamma_\tau} \mid \eta < \tau\rangle$ is a $\leq^0_{\gamma_\tau}$-decreasing sequence in $(\mathring{\mathbb{P}}_{\gamma_\tau})_n$. Thus, by Lemma 3.13, we may find a lower bound b_τ in $(\mathring{\mathbb{P}}_{\gamma_\tau})_n$ such that $B_{b_\tau} = \bigcup_{\eta<\tau} B_{b_\eta}$. In effect, $b_\tau \in D^\epsilon_{\gamma_\tau,n}$.

At the end of the above process, we have obtained $b_{\mathrm{cf}(\delta)}$ which is an element of $D^\epsilon_{\delta,n}$ extending a, as desired. □

For each $n < \omega$, let us say that $\dagger_{\delta,n}$ holds iff, for all $\epsilon < \mu$, $D^\epsilon_{\delta,n}$ is dense in $(\mathbb{P}_\delta)_n$. By Claim 4.24.1, $\dagger_{\delta,n}$ holds for all $n \geq m$. In particular, if $m = 0$, then we are done. To address the general case, we now assume that we are given $n < \omega$ such that $\dagger_{\delta,n+1}$ holds, and we shall prove that $\dagger_{\delta,n}$ holds, as well.

Let $\epsilon < \mu$, and let $a \in P_\delta$ with $\ell_\delta(a) = n$; we need to find a condition $b \in D^\epsilon_{\delta,n}$ extending a. To this end, we shall first construct a $\leq^0_\delta$-decreasing sequence $\langle a_j \mid j < \omega\rangle$ and an increasing sequence of ordinals $\langle \epsilon_j \mid j < \omega\rangle$ such that, for each $j < \omega$ and $\beta + 1 \in B_{a_j} \setminus 2$, all of the following will hold:

(I) For all $q \in W_1(a_{j+1} \restriction \beta)$, $\sigma^{a_j} \leq \epsilon_j < \sigma^{a_{j+1},\beta}(q)$;

(II) For all $q \in W_{\geq 1}(a_{j+1})$, $q \restriction \beta \Vdash_{(\mathbb{P}_\beta)_{\ell_\delta(q)}} \dot{C}^\beta_{\ell_\delta(q)} \cap (\check{\epsilon}_j, \check{\epsilon}_{j+1}) \neq \emptyset$.

The construction of $\langle (a_j, \epsilon_j) \mid j < \omega\rangle$ is by recursion on $j < \omega$. We start by setting $(a_0, \epsilon_0) := (a, \max\{\sigma^a, \epsilon\})$. Next, suppose that $j < \omega$ and that the pair (a_j, ϵ_j) has already been defined. Since $\dagger_{\delta,n+1}$ holds, $D^{\epsilon_j}_{\delta,n+1}$ is a set of conditions in $(\mathring{\mathbb{P}}_\delta)_{n+1}$ which is dense in $(\mathbb{P}_\delta)_{n+1}$. Let $\vec{s} = \langle s_\xi \mid \xi < \chi\rangle$ be a good enumeration of $W_1(a_j)$. By Lemma 3.11, we may now use the winning strategy of **I** in playing the game $\Game_{\mathbb{P}_\delta}(a_j, \vec{s}, D^{\epsilon_j}_{\delta,n+1})$, thus obtaining a sequence $\langle b_\xi \mid \xi < \chi\rangle$ of conditions in $D^{\epsilon_j}_{\delta,n+1}$ along with a condition $a_{j+1} \leq^0_\delta a_j$ such that a_{j+1} diagonalizes $\langle b_\xi \mid \xi < \chi\rangle$ with respect to $\vec{s}$. Finally, set $\epsilon_{j+1} := \sup_{\xi<\chi} \sigma^{b_\xi}$.

The reason we diagonalize conditions in $D^{\epsilon_j}_{\delta,n+1}$ will become apparent in the verification of Clauses (I) and (II):

Claim 4.24.2. *For each β with $\beta+1 \in B_{a_j} \setminus 2$ and every $q \in W_1(a_{j+1} \restriction \beta)$, $\sigma^{a_j} \leq \epsilon_j < \sigma^{a_{j+1},\beta}(q)$.*

Proof. Suppose that β and q are as above. Defining $c := \Cap_{\delta,\beta}(a_{j+1})(q)$ we have that $c \in W_1(a_{j+1})$, and so there is $\xi < \chi$ such that $c \leq^0_\delta b_\xi$. Since $b_\xi \leq_\delta a_j$, $\beta+1 \in B_{a_j} \subseteq B_{b_\xi}$. Also, $(c \restriction \beta+1) \leq^0_{\beta+1} (b_\xi \restriction \beta+1)$ and $c \restriction \beta = q$, so $w(b_\xi \restriction \beta, c \restriction \beta) = w(b_\xi \restriction \beta, q) = b_\xi \restriction \beta$. Thus, Clauses (2) and (3) of Lemma 4.23 together yield

$$\epsilon_j < \sigma^{b_\xi,\beta}(b_\xi \restriction \beta) \leq \sigma^{c,\beta}(q) = \sigma^{a_{j+1},\beta}(w(a_{j+1} \restriction \beta, q)) = \sigma^{a_{j+1},\beta}(q),$$

where the rightmost equality follows from $q \in W(a_{j+1} \restriction \beta)$.

Finally, since $b_0 \leq_\delta a_j$, Lemma 4.23(3) yields $\sigma^{a_j} \leq \sigma^{b_0} \leq \epsilon_j$. □

Claim 4.24.3. *For each β with $\beta+1 \in B_{a_j} \setminus 2$ and every $q \in W_{\geq 1}(a_{j+1})$,*

$$q \restriction \beta \Vdash_{(\mathbb{P}_\beta)_{\ell_\delta(q)}} \dot{C}^\beta_{\ell_\delta(q)} \cap (\check{\epsilon}_j, \check{\epsilon}_{j+1}) \neq \emptyset.$$

Proof. Suppose that β and q are as above. Appealing to Fact 2.6, let $\tilde{q}$ be the unique member of $W_1(a_{j+1})$ such that $q \leq_\delta \tilde{q}$.

Since a_{j+1} diagonalizes $\langle b_\xi \mid \xi < \chi\rangle$, there is $\xi < \chi$ such that $\tilde{q} \leq^0_\delta b_\xi$. Note that $\beta+1 \in B_{b_\xi} \setminus 2$, as $b_\xi \leq_\delta a_j$. Also, since $b_\xi \in D^{\epsilon_j}_{\delta,n+1} \subseteq \mathring{P}_\delta$, Lemma 3.9 yields $(b_\xi \restriction \beta+1) \in \mathring{P}_{\beta+1}$. In particular, $\mathrm{mtp}_{\beta+1}(b_\xi \restriction \beta+1) = 0$. Equivalently,

$$(\sigma^{b_\xi,\beta}(q), b_\xi \restriction \beta) \in R_\beta \text{ for all } q \in W(b_\xi \restriction \beta).^{27}$$

Since $q \restriction \beta \leq_\beta \tilde{q} \restriction \beta \leq^0_\beta b_\xi \restriction \beta$, $(\sigma^{b_\xi,\beta}(b_\xi \restriction \beta), q \restriction \beta) \in R_\beta$. Therefore,

$$q \restriction \beta \Vdash_{(\mathbb{P}_\beta)_{\ell_\delta(q)}} \sigma^{b_\xi,\beta}(b_\xi \restriction \beta) \in \dot{C}^\beta_{\ell_\delta(q)}.$$

Finally, as $\epsilon_j < \sigma^{b_\xi,\beta}(b_\xi \restriction \beta) \leq \sup_{\xi<\chi} \sigma^{b_\xi} < \epsilon_{j+1}$, the claim follows. □

This produces a sequence $\langle (a_j, \epsilon_j) \mid j < \omega\rangle$ witnessing Clauses (I) and (II), with $(a_0, \epsilon_0) = (a, \epsilon)$. Set $\epsilon_\omega := \sup_{j<\omega} \epsilon_j$.

Claim 4.24.4. *There is $b \in \mathring{P}_\delta$ such that $b \leq^0_\delta a_j$ for all $j < \omega$.*

In particular, there is $b \in D^\epsilon_{\delta,n}$ such that $b \leq^0_\delta a$.

Proof. Let $\langle \gamma_\tau \mid \tau < \theta\rangle$ be the increasing enumeration of $\bigcup_{j<\omega} B_{a_j} \setminus 2$.

The sought condition b will be obtained as the union of a sequence of conditions $\langle b_\tau \mid \tau < \theta\rangle$ for which the following hold:

(a) $b_\tau \in (\mathring{P}_{\gamma_\tau})$ and $B_{b_\tau} \setminus 2 = \{\gamma_\varrho \mid \varrho \leq \tau\}$;
(b) $\sigma^{b_\tau,\beta}(b_\tau \restriction \beta) = \epsilon_\omega$ for all $\beta+1 \in B_{b_\tau}$;
(c) $b_\tau \leq^0_{\gamma_\tau} a_j \restriction \gamma_\tau$ for all $j < \omega$;
(d) $b_\tau \restriction \gamma_\varrho = b_\varrho$ for all $\varrho \leq \tau$.

For each $\tau < \theta$, we shall denote by β_τ the predecessor of γ_τ.

Case $\tau = 0$: Since $\langle a_j \restriction 1 \mid n < \omega\rangle$ is a decreasing sequence of conditions in $(\mathbb{P}_1)_n = (\mathring{\mathbb{P}}_1)_n$, Definition 2.3(2) yields a $\leq^0_1$-lower bound $p \in (P_1)_n$ for it.[28]

For each $j < \omega$, set

$$c_j := \Cap_{\gamma_0,\beta_0}(a_j \restriction \gamma_0)(\lceil p \rceil^{\mathbb{P}_{\beta_0}}).$$

Since $\lceil p \rceil^{\mathbb{P}_{\beta_0}} \leq^0_{\beta_0} \lceil a_j \restriction 1 \rceil^{\mathbb{P}_{\beta_0}}$ it follows that $c_j \leq^0_{\gamma_0} a_j \restriction \gamma_0$ for each $j < \omega$. Actually, by Clauses (3) and (6) of Lemma 3.5, the sequence $\langle c_j \mid j < \omega\rangle$ is $\leq^{\pi_{\gamma_0,\beta_0}}_{\gamma_0}$-decreasing hence, as in the proof of [PRS20, Lemma 6.15], it is order-isomorphic to $(\omega, \ni)$.

[27] See Definition 4.5(4) and Definition 4.13.

[28] The equality $(\mathbb{P}_1)_n = (\mathring{\mathbb{P}}_1)_n$ was part of Building Block I.

Let $j_0 < \omega$ denote the least index such that $\gamma_0 \in B_{a_j}$ for each $j \geq j_0$. For each $j \geq j_0$, put $\vec{R}^j := \vec{S}^{c_j,\beta_0}$ so that $c_j = \lceil p \rceil^{\mathbb{P}_{\beta_0}} {}^\frown \langle \vec{R}^j \rangle$. By Definition 4.9(*),

$$\max(\mathrm{dom}(\vec{R}^j)) = \max(\mathrm{dom}(\vec{S}^{a_j,\beta_0})) = \alpha^{a_j,\beta_0},$$

and

$(\star)$ $\quad R^j_i(q) = S^{a_j,\beta_0}_i(w(a_j \restriction \beta_0, q))$ for all $i \leq \alpha^{a_j,\beta_0}$ and $q \in W(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$.

Set $\alpha := \sup_{j_0 \leq j < \omega} \alpha_j$, where $\alpha_j := \alpha^{a_j,\beta_0}$ for each $j \geq j_0$. Let $b_0 := \lceil p \rceil^{\mathbb{P}_{\beta_0}} {}^\frown \langle \vec{S} \rangle$, where $\vec{S}$ is the sequence $\langle S_i \mid i \leq \alpha + 1 \rangle$ defined according to the following casuistic:

▶ For $i < \alpha$, $S_i(q)$ is defined as the unique member of

$$\{R^j_i(q) \mid j \geq j_0,\, \alpha_j \geq i\}.$$

▶ For $i = \alpha$, we distinguish several cases:
▶▶ If $\alpha = \alpha_j$, for some $j \geq j_0$, then $S_\alpha(q) := R^j_{\alpha_j}(q)$;
▶▶ If $S_i(q) = \emptyset$ for all $i < \alpha$, then we continue and let $S_\alpha(q) := \emptyset$;
▶▶ Otherwise, set $S_\alpha(q) := \bigcup_{i<\alpha} S_i(q) \cup \{\varepsilon_q\}$, where

$$\varepsilon_q := \sup\{\max(S_i(q)) \mid i < \alpha,\, S_i(q) \neq \emptyset\}.$$

▶ For $i = \alpha + 1$, let $S_{\alpha+1}(q) := S_\alpha(q) \cup \{\epsilon_\omega\}$.

A moment's reflection makes it clear that

$$\varepsilon_q = \sup\{\max(R^j_{\alpha_j}(q)) \mid j \geq j_0,\, R^j_{\alpha_j}(q) \neq \emptyset\}.$$

Subclaim 4.24.4.1. *b_0 witnesses Clauses (a)–(d).*

Proof. Note that Clause (d) is trivially true at this stage. Assume for a moment that Clause (a) holds. By assumption, b_0 is a condition in $\mathbb{P}_{\gamma_0}$ and by construction $b_0 \leq^0_{\gamma_0} c_j \leq^0_{\gamma_0} a_j \restriction \gamma_0$ for all $j < \omega$. Thus, Clause (c) holds.

For Clause (b), note that $B_{b_0} \setminus 2 = \{\gamma_0\}$ and

$$\sigma^{b_0,\beta_0}(b_0 \restriction \beta_0) = \sigma^{b_0,\beta_0}(\lceil p \rceil^{\mathbb{P}_{\beta_0}}) = \epsilon_\omega.$$

Thus we are left with verifying the first assertion of Clause (a).

Let us first prove that $b_0 \in P_{\gamma_0}$, which amounts to check that $\vec{S}$ is a $\lceil p \rceil^{\mathbb{P}_{\beta_0}}$-strategy. Since for each $j < \omega$, $\vec{R}^j$ is a $\lceil p \rceil^{\mathbb{P}_{\beta_0}}$-strategy a moment of reflection makes clear that we just need to verify that both S_α and $S_{\alpha+1}$ are labeled $\lceil p \rceil^{\mathbb{P}_{\beta_0}}$-trees. Actually, since $\langle S_i(q) \mid i < \alpha \rangle$ is a weakly $\sqsubseteq$-increasing sequence of closed sets it is enough to check Clauses (3) and (4) of Definition 4.5. We commence checking this for S_α. The proof for $S_{\alpha+1}$ will be straightforward once we are done with that for S_α.

Clause (3) for S_α: Fix $q \in W(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$ and to avoid trivialities assume that $S_\alpha(q) \neq \emptyset$. For each $j \geq j_0$, Clause (3) for $\vec{R}^j$ yields

$$q \Vdash_{\mathbb{P}_{\gamma_0}} R^j_{\alpha_j}(q) \cap (\dot{T}^+)^{\beta_0} = \emptyset,$$

hence it is enough with discussing the case where $S_\alpha(q) = (\bigcup_{i<\alpha} S_i(q)) \cup \{\varepsilon_q\}$. To establish the clause it will be enough to show that $q \Vdash_{\mathbb{P}_{\gamma_0}} \varepsilon_q \notin (\dot{T}^+)^{\beta_0}$.

Set $\vec{\sigma} := \langle \sigma_j \mid j \geq j_0 \rangle$, where $\sigma_j := \sigma^{c_j,\beta_0}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$ for each $j \geq j_0$.

We claim that:

$$\varepsilon_q = \begin{cases} \max(\mathrm{Im}(\vec{\sigma})), & \text{if } q = \lceil p \rceil^{\mathbb{P}_{\beta_0}} \ \&\ \vec{\sigma} \text{ is eventually constant;} \\ \epsilon_\omega, & \text{otherwise.} \end{cases}$$

Case 1: By the definition of $\vec{S}$, for each $j \geq j_0$,

$$\max(S_{\alpha_j}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})) = \max(R^j_{\alpha_j}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})) = \sigma_j.$$

Also, since $\vec{\sigma}$ is eventually constant there is $j^\star \geq j_0$ such that

$$\max(R^j_{\alpha_j}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})) = \sigma_{j^\star} \text{ for all } j \geq j^\star.$$

So, for each $j \geq j^\star$, Clause (3) of Definition 4.7 for $\vec{R}^{j+1}$ yields

$$S_{\alpha_j}(\lceil p \rceil^{\mathbb{P}_{\beta_0}}) = S_{\alpha_{j+1}}(\lceil p \rceil^{\mathbb{P}_{\beta_0}}).$$

Thus, $S_\alpha(\lceil p \rceil^{\mathbb{P}_{\beta_0}}) = S_{\alpha_{j^\star}}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$. Therefore, $\varepsilon_q = \sigma_{j^\star} = \max(\vec{\sigma})$.

Case 2: Here we need to discuss two cases:

▶ Suppose that $q \in W_{\geq 1}(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$. Using Fact 4.2, let $\tilde{q} \in W_1(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$ be such that $q \leq_{\beta_0} \tilde{q}$. Combining $(\star)$ with Clause (I) we have

$$\max(R^j_{\alpha_j}(q)) \leq \sigma^{a_j} \leq \epsilon_{j+1} \text{ for all } j \geq j_0.$$

On the other hand, by Clause (I) and Lemma 4.23(1)

$$\epsilon_j \leq \sigma^{a_{j+1},\beta_0}(w(a_{j+1} \restriction \beta_0, \tilde{q})) \leq \sigma^{a_{j+1},\beta_0}(w(a_{j+1} \restriction \beta_0, q)).$$

Also, recalling how c_{j+1} was defined, Lemma 4.23(2) yields

$$\sigma^{a_{j+1},\beta_0}(w(a_{j+1} \restriction \beta_0, q)) = \sigma^{c_{j+1},\beta_0}(q) = \max(R^{j+1}_{\alpha_{j+1}}(q)).$$

Thus, $\epsilon_j \leq \max(R^{j+1}_{\alpha_{j+1}}(q))$. Combining the above we get $\varepsilon_q = \epsilon_\omega$, as wanted.

▶ Suppose that $\vec{\sigma}$ is not eventually constant and that $q = \lceil p \rceil^{\mathbb{P}_{\beta_0}}$.

Let $j \geq j_0$ non-zero such that $\sigma_j < \sigma_{j+1}$. Our first task is to prove that

$$(\star\star) \ \ \sigma^{a_j,\beta_0}(w(a_j \restriction \beta_0, r)) < \sigma_{j+1}, \text{ for all } r \in W_1(a_{j+1} \restriction \beta_0).$$

Note that by equation $(\star)$ above we have

$$\max(R^j_{\alpha_j}(r)) = \max(S^{a_j,\beta_0}_{\alpha_j}(w(a_j \restriction \beta_0, r))) = \sigma^{a_j,\beta_0}(w(a_j \restriction \beta_0, r)),$$

hence all we need to prove is $\sigma_{j+1} > \max(R^j_{\alpha_j}(r))$. Once we establish $(\star\star)$ we will be able to show that $\varepsilon_q = \epsilon_\omega$.

So, fix $r \in W_1(a_{j+1} \restriction \beta_0)$ and let us look at the sequence

$$\langle \max(R^{j+1}_i(q)) \mid \alpha_j \leq i \leq \alpha_{j+1} \rangle.$$

By Definition 4.7(3) for $\vec{R}^{j+1}$ the above sequence is weakly increasing. Also, note that the the first value of this sequence is σ_j and the last one is σ_{j+1}.

Let $i^\star \leq \alpha_{j+1}$ be the first index such that $\max(R_i^{j+1}(q)) = \sigma_{j+1}$ for all $i \in [i^\star, \alpha^{a_{j+1},\beta_0}]$. Since by assumption $\sigma_j < \sigma_{j+1}$ note that $\alpha_j < i^\star$.

►► If $i^\star$ takes the form $k+1$ then Definition 4.7(4) for $\vec{R}^{j+1}$ yields

$$R_{i^\star}^{j+1}(q) \setminus R_k^{j+1}(q) \sqsubseteq R_{i^\star}^{j+1}(r) \setminus R_k^{j+1}(r).$$

By minimality of $i^\star$, $\sigma_{j+1} = \max(R_{i^\star}^{j+1}(q)) > \max(R_k^{j+1}(q))$. In particular, σ_{j+1} is a member of the left-hand-side of the above expression and thus $\sigma_{j+1} > \max(R_k^{j+1}(r))$. Since $\alpha_j < i^\star$, then $\alpha_j \leq k$ and so

$$\sigma_{j+1} > \max(R_k^{j+1}(r)) \geq \max(R_{\alpha_j}^{j+1}(r)) = \max(R_{\alpha_j}^{j}(r)).$$

►► If $i^\star$ is limit then Clause (5) of Definition 4.7 yields

$$R_{i^\star}^{j+1}(q) = \bigcup_{k<i^\star} R_k^{j+1}(q) \cup \{\sigma_{j+1}\}.$$

By minimality of $i^\star$, unboundedly many $k \in (\alpha_j, i^\star)$ must satisfy

$$\max(R_k^{j+1}(q)) < \max(R_{k+1}^{j+1}(q)) < \sigma_{j+1}.$$

Thus, again by Clauses (3) and (4) of Definition 4.7 for $\vec{R}^{j+1}$,

$$\max(R_{\alpha_j}^{j}(r)) = \max(R_{\alpha_j}^{j+1}(r)) \leq \max(R_k^{j+1}(r)) < \max(R_{k+1}^{j+1}(q)) < \sigma_{j+1}.$$

The above discussion yields $(\star\star)$.

Combining $(\star\star)$ with Clause (I) for a_j we get

$$\epsilon_{j-1} < \sigma^{a_j,\beta_0}(w(a_j \restriction \beta_0, r)) < \sigma_{j+1}.$$

On the other hand, equation $(\star)$ yields

$$R_{\alpha_{j+1}}^{j+1}(q) = S_{\alpha_{j+1}}^{a_{j+1},\beta_0}(w(a_{j+1} \restriction \beta_0, q)),$$

hence Clause (I) implies that $\sigma_{j+1} \leq \sigma^{a_{j+1}} \leq \epsilon_{j+1}$. Altogether,

$$\epsilon_\omega = \sup_{j \geq j_0} \sigma_j = \varepsilon_q.^{29}$$

After computing ε_q we are in conditions to show that $q \Vdash_{\mathbb{P}_{\gamma_0}} \varepsilon_q \notin (\dot{T}^+)^{\beta_0}$.

The case in which $q = \lceil p \rceil^{\mathbb{P}_0}$ and $\vec{\sigma}$ is eventually constant is trivial. Indeed, let $j \geq j_0$ be such that $\max(\vec{\sigma}) = \sigma_j$. By the above discussion, $\varepsilon_q = \sigma_j$. Appealing to Clause (2) of Definition 4.5 for R^j we have $\sigma_j \in R_{\alpha_j}^j(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$, hence $\varepsilon_q \in R_{\alpha_j}^j(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$. Finally, Clause (3) of Definition 4.5 for $R_{\alpha_j}^j$ yields $q \Vdash_{\mathbb{P}_{\gamma_0}} R_{\alpha_j}^j(q) \cap (\dot{T}^+)^{\beta_0} = \emptyset$ and thus $q \Vdash_{\mathbb{P}_{\gamma_0}} \varepsilon_q \notin (\dot{T}^+)^{\beta_0}$.

Otherwise, the previous discussion yields $\varepsilon_q = \epsilon_\omega$. In this case, instead of proving $q \Vdash_{\mathbb{P}_{\gamma_0}} \varepsilon_q \notin (\dot{T}^+)^{\beta_0}$ we will show more: namely, we will prove that $(\epsilon_\omega, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$. This will become handy when verifying Clause (4) for S_α and Clauses (3) and (4) for $S_{\alpha+1}$.

[29]For the latter equality recall the expression displayed right before Subclaim 4.24.4.1.

Recalling the paragraph preceding Definition 4.5, $(\epsilon_\omega, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$ amounts to say that $r \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(r)}} \epsilon_\omega \in \dot{C}^{\beta_0}_{\ell_{\beta_0}(r)}$ for all $r \leq_{\beta_0} \lceil p \rceil^{\mathbb{P}_{\beta_0}}$.[30] Fix $r \leq_{\beta_0} \lceil p \rceil^{\mathbb{P}_{\beta_0}}$.

▶ If $r \in (P_{\beta_0})^{\lceil p \rceil^{\mathbb{P}_{\beta_0}}}_{\geq 1}$ then, for each $j \geq j_0$, $\Cap_{\delta,\beta_0}(a_{j+1})(r) \in (P_\delta)^{a_{j+1}}_{\geq 1}$ and so there is some $c_{j+1} \in W_{\geq 1}(a_{j+1})$ such that $\Cap_{\delta,\beta_0}(a_{j+1})(r) \leq^0_\delta c_{j+1}$. By Lemma 3.5(3), $r \leq^0_{\beta_0} c_{j+1} \restriction \beta_0$. Observe that $\gamma_0 \in B_{a_j}$, hence Clause (II) for a_{j+1} yields

$$c_{j+1} \restriction \beta_0 \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(r)}} \dot{C}^{\beta_0}_{\ell_{\beta_0}(r)} \cap (\check{\epsilon}_j, \check{\epsilon}_{j+1}) \neq \emptyset.$$

Consequently, for each $j \geq j_0$, $r \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(r)}} \dot{C}^{\beta_0}_{\ell_{\beta_0}(r)} \cap (\check{\epsilon}_j, \check{\epsilon}_{j+1}) \neq \emptyset$.

Finally, since $r \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(r)}}$ "$\dot{C}^{\beta_0}_{\ell_{\beta_0}(r)}$ is a club" one concludes that

$$r \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(r)}} \check{\epsilon}_\omega \in \dot{C}^{\beta_0}_{\ell_{\beta_0}(r)}. \tag{$\mathcal{R}$}$$

▶ Otherwise, $r \in (P_{\beta_0})^{\lceil p \rceil^{\mathbb{P}_{\beta_0}}}_0$. Then, we first claim that

$$\lceil p \rceil^{\mathbb{P}_{\beta_0}} \Vdash_{\mathbb{P}_{\beta_0}} \check{\epsilon}_\omega \notin (\dot{T}^+)^{\beta_0}.$$

Let $s \in (P_{\beta_0})^{\lceil p \rceil^{\mathbb{P}_{\beta_0}}}_{\geq 1}$ such that $s \Vdash_{\mathbb{P}_{\beta_0}} \check{\epsilon}_\omega \in (\dot{T}^+)^{\beta_0}$. Clearly, this yields $s \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(s)}} \check{\epsilon}_\omega \in (\dot{T}^+)^{\beta_0}_{\ell_{\beta_0}(s)}$, and so $s \Vdash_{(\mathbb{P}_{\beta_0})_{\ell_{\beta_0}(s)}} \check{\epsilon}_\omega \notin \dot{C}^{\beta_0}_{\ell_{\beta_0}(s)}$. Note that this produces a contradiction with equation ($\mathcal{R}$) above.

So, $\lceil p \rceil^{\mathbb{P}_{\beta_0}} \Vdash_{\mathbb{P}_{\beta_0}} \check{\epsilon}_\omega \notin (\dot{T}^+)^{\beta_0}$. In particular, $\lceil p \rceil^{\mathbb{P}_{\beta_0}} \Vdash_{(\mathbb{P}_{\beta_0})_n} \check{\epsilon}_\omega \notin (\dot{T}^+)^{\beta_0}_n$, hence $\lceil p \rceil^{\mathbb{P}_{\beta_0}} \Vdash_{(\mathbb{P}_{\beta_0})_n} \check{\epsilon}_\omega \in \dot{C}^{\beta_0}_n$.[31] In effect, the same is $(\mathbb{P}_{\beta_0})_n$-forced by r.

Therefore, we have proved that $(\varepsilon_q, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$ for all $q \in W(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$.

Note that at this point we have managed to argue that $q \Vdash_{\mathbb{P}} \varepsilon_q \notin (\dot{T}^+)^{\beta_0}$ for all $q \in W(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$. This completes the proof of Clause (3) for S_α.

Clause (4) for S_α: Let $q' \leq_{\mathbb{P}_{\gamma_0}} q$ be a pair of members in $W(\lceil p \rceil^{\mathbb{P}_{\beta_0}})$ with $\ell(q) \geq \ell(\lceil p \rceil^{\mathbb{P}_{\beta_0}}) + 1$. Then $q' \neq \lceil p \rceil^{\mathbb{P}_{\beta_0}}$ and so the previous discussion yields $\varepsilon_q = \epsilon_\omega$ and $(\epsilon_\omega, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$, hence $(\epsilon_\omega, q) \in R_{\beta_0}$, as well.

Clauses (3) & (4) of $S_{\alpha+1}$: By the preceding discussion it is clear that $\max(S_{\alpha+1}(q)) = \epsilon_\omega$.Also, we have proved that $(\epsilon_\omega, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$. Arguing very similar as above one can use this to prove Clauses (3) and (4) for $S_{\alpha+1}$. Moreover, regarding Clause (4), $(\epsilon_\omega, \lceil p \rceil^{\mathbb{P}_{\beta_0}}) \in R_{\beta_0}$ yields $m(S_{\alpha+1}) = 0$.

To complete the proof of Clause (a) we still need to argue that $b_0 \in \mathring{P}_{\gamma_0}$. By the above proof, $b_0 \in P_{\gamma_0}$ and $\mathrm{mtp}_{\gamma_0}(b_0) = m(S_{\alpha+1}) = 0$. Also, since

[30] Here note that we are (crucially) using that in Building Block II we appeal to Corollary 4.18 with respect to the condition $\mathbf{1}_{\mathbb{P}_{\beta_0}}$.

[31] Recall Remark 4.21.

$p \in \mathring{P}_1$ and $\lceil p \rceil^{\mathbb{P}_{\beta_0}} = p * \emptyset_{\beta_0}$, we may appeal to Lemma 3.9 and infer that $b_0 \restriction \beta_0 = \lceil p \rceil^{\mathbb{P}_{\beta_0}} \in \mathring{P}_{\beta_0}$. So, recalling Definition 3.8, we have $b_0 \in \mathring{P}_{\gamma_0}$. □

This completes the construction when $\tau = 0$.

Case $\tau > 0$: Suppose that $\langle b_\eta \mid \eta < \tau \rangle$ has been already defined. Put, $b^*_\tau := (\bigcup_{\eta<\tau} b_\eta) * \emptyset_{\beta_\tau}$ and for each $j < \omega$, set $c^\tau_j := \pitchfork_{\gamma_\tau,\beta_\tau}(a_j \restriction \gamma_\tau)(b^*_\tau)$.

Subclaim 4.24.4.2. *$b^*_\tau \in \mathring{P}_{\beta_\tau}$ and $b^*_\tau \leq^0_{\beta_\tau} a_j \restriction \beta_\tau$ for all $j < \omega$.*

Proof. Note that once we establish the former assertion the latter will follow automatically from Clauses (c) and (d) of our induction hypothesis.

▶ If τ takes the form $\bar{\eta} + 1$ then Clauses (a) and (d) of the induction hypothesis yield $\bigcup_{\eta<\tau} b_\eta = b_{\bar{\eta}} \in \mathring{P}_{\gamma_{\bar{\eta}}}$. Thus, Lemma 3.9 yields $b^*_\tau \in \mathring{P}_{\beta_\tau}$.

▶ Otherwise, set $\bar{\gamma} = \sup_{\eta<\tau} \gamma_\eta$ and note that $\bar{\gamma} \leq \beta_\tau$. By Clause (d) of the induction hypothesis, $\bar{b} := \bigcup_{\eta<\tau} b_\eta$ is a condition in $\mathbb{P}_{\bar{\gamma}}$. If we show that $\bar{b} \in P_{\bar{\gamma}}$ then Lemma 3.9 will imply that $b^*_\tau \in \mathring{P}_{\beta_\tau}$ and we will be done.

Let $\beta \in B_{\bar{b}} \cup \{1\}$ and $\eta < \tau$ be such that $\beta \in B_{b_\eta}$. By Clause (d) of the induction hypothesis, $\bar{b} \restriction \beta = b_\tau \restriction \beta$, hence combining Clause (a) for b_τ with Lemma 3.9 we get $\bar{b} \restriction \beta \in \mathring{P}_\beta$. Since $\bar{\gamma}$ is limit then $\bar{b} \in \mathring{P}_{\bar{\gamma}}$. □

By the above result, $c^\tau_j \in P_{\gamma_\tau}$ and $c^\tau_j \leq^0_{\gamma_\tau} a_j \restriction \gamma_\tau$ for each $j < \omega$. Actually, $\langle c^\tau_j \mid j < \omega \rangle$ is a $\leq^{\pi_{\gamma_\tau,\beta_\tau}}_{\gamma_\tau}$-decreasing sequence so, as before, it is order-isomorphic to $(\omega, \ni)$. As in the preceding argument, denote by j_τ the least index such that $\gamma_\tau \in B_{a_j}$ for all $j \geq j_\tau$. For each $j \geq j_\tau$, put $\vec{R}^{j,\tau} := \vec{S}^{c_j,\beta_\tau}$ so that $c^\tau_j = b^{*\frown}_\tau \langle \vec{R}^{j,\tau} \rangle$. Set $\alpha := \sup_{j_\tau \leq j < \omega} \alpha_j$, where $\alpha_j := \alpha^{a_j,\beta_\tau}$ for each $j \geq j_\tau$. We define $b_\tau := b^{*\frown}_\tau \langle \vec{S} \rangle$, where $\vec{S}$ is the sequence $\langle S_i \mid i \leq \alpha + 1 \rangle$ defined exactly as in the case $\tau = 0$ above replacing $\vec{R}^j$ by $\vec{R}^{j,\tau}$.

We claim that b_τ witnesses Clauses (a)–(d): Arguing as in the case where $\tau = 0$ we get $b_\tau \in P_{\gamma_\tau}$ and $\mathrm{mtp}_{\gamma_\tau}(b_\tau) = m(S_{\alpha+1}) = 0$. Additionally, by Subclaim 4.24.4.2, $b_\tau \restriction \beta_\tau = b^*_\tau \in \mathring{P}_{\beta_\tau}$, hence $b_\tau \in \mathring{P}_{\gamma_\tau}$. Finally,

$$B_{b_\tau} \setminus 2 = (B_{b^*_\tau} \setminus 2) \cup \{\gamma_\tau\} = \{\gamma_\eta \mid \eta \leq \tau\},$$

where the second equality comes from Clause (a) of the induction hypothesis.

For the verification of Clause (b), let us fix $\eta \leq \tau$. If $\eta = \tau$ then one argues as in the case where $\tau = 0$ that $\sigma^{b_\tau,\beta_\tau}(b_\tau \restriction \beta_\tau) = \sigma^{b_\tau,\beta_\tau}(b^*_\tau) = \epsilon_\omega$. Otherwise, if $\eta < \tau$, Clauses (b) and (d) of the induction hypothesis yield

$$\sigma^{b_\tau,\beta_\eta}(b_\tau \restriction \beta_\eta) = \sigma^{b_\eta,\beta_\eta}(b_\eta \restriction \beta_\eta) = \epsilon_\omega.$$

Clause (c) follows noting that $b_\tau \leq^0_{\gamma_\tau} c_j \leq^0_{\gamma_\tau} a_j \restriction \gamma_\tau$ for all $j < \omega$. Finally, Clause (d) for b_τ is a consequence of $b_\tau \restriction \beta_\tau = b^*_\tau$ and Clause (d) of the induction hypothesis.

The above discussion completes the construction of a sequence $\langle b_\tau \mid \tau < \theta \rangle$ witnessing Clauses (a)–(d) above. Finally, set $b := (\bigcup_{\tau<\theta} b_\tau) * \emptyset_\delta$.

Subclaim 4.24.4.3. *$b \in D^\epsilon_{n,\delta}$ and $b \leq^0_\delta a_j$ for all $j < \omega$.*

Proof. By Clause (d) we have that $b \in P_\delta$, hence the latter assertion of the claim is an outright consequence of Clauses (a) and (c).

So, we are left with checking that $b \in D^\epsilon_{n,\delta}$. Let $\beta \in B_b \setminus 2$ and $\tau < \theta$ such that $\beta = \gamma_\tau$. Appealing to Clause (d) for $b_{\tau+1}$, $b \restriction \beta = b_{\tau+1} \restriction \beta = b_\tau$, hence

$$\sigma^{b,\beta}(b \restriction \beta) = \sigma^{b_\tau,\beta_\tau}(b_\tau \restriction \beta_\tau) = \epsilon_\omega > \epsilon.$$

Also, by Clause (a) for b_τ, $b \restriction \beta \in \mathring{P}_\beta$. Likewise, by Lemma 3.9, $b \restriction 1 \in \mathring{P}_1$. Altogether, we have proved that $b \in D^\epsilon_{n,\delta}$ as desired. □

The above completes the proof of Claim 4.24.4. □

This finishes the proof of Lemma 4.24. □

Thanks to Lemma 4.24 we can now appeal to the iteration scheme of Section 3 with respect to the building blocks of this section and obtain, in return, a Σ-Prikry triple $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$.

Theorem 4.25. *In $V^{\mathbb{P}_{\mu^+}}$ all of the following hold true:*

(1) Any cardinal in V remains a cardinal and retains its cofinality;
(2) κ is a singular strong limit of countable cofinality;
(3) $2^\kappa = \kappa^{++}$;
(4) $\mathrm{Refl}(<\omega, \kappa^+)$.

Proof. (1) By Fact 2.7(1), no cardinal $\leq \kappa$ changes its cofinality; by Fact 2.7(3), κ^+ is not collapsed, and by Definition 2.3(3), no cardinal $> \kappa^+$ changes its cofinality.

(2) In V, κ is a singular strong limit of countable cofinality, and so by Fact 2.7(1), this remains valid in $V^{\mathbb{P}_{\mu^+}}$.

(3) In V, we have that $2^\kappa = \kappa^+$. In addition, by Remark 3.3(1), $\mathbb{P}_{\mu^+}$ is isomorphic to a subset of H_{μ^+}, so that, from $|H_{\mu^+}| = \kappa^{++}$, we infer that $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \leq \kappa^{++}$. Finally, as $\mathbb{P}_{\mu^+}$ projects to $\mathbb{P}_1$ which is isomorphic to $\mathbb{Q}$, we get that $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \geq \kappa^{++}$. Altogether, $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa = \kappa^{++}$.

(4) As $\kappa^+ = \mu$ and κ is singular, $\mathrm{Refl}(<\omega, \kappa^+)$ is equivalent to $\mathrm{Refl}(<\omega, E^\mu_{<\kappa})$. By Fact 4.19, we already know that $V^{\mathbb{P}_{\mu^+}} \models \mathrm{Refl}(<\omega, \Gamma)$. So, by Proposition 4.3, it suffices to verify that $\mathrm{Refl}(<2, (E^\mu_\omega)^V, \Gamma)$ holds in $V^{\mathbb{P}_{\mu^+}}$.

Let G be $\mathbb{P}_{\mu^+}$-generic over V and hereafter work within $V[G]$. Towards a contradiction, suppose that there exists a subset T of $(E^\mu_\omega)^V$ that does not reflect in Γ. Fix $r^* \in G$ and a $\mathbb{P}_{\mu^+}$-name τ such that τ_G is equal to such a T and such that r^* forces τ to be a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ. Furthermore, we may require that τ be a *nice name*, i.e., each element of τ is a pair $(\check{\xi}, p)$ where $(\xi, p) \in (E^\mu_\omega)^V \times P_{\mu^+}$, and, for all $\xi \in (E^\mu_\omega)^V$, the set $\{p \mid (\check{\xi}, p) \in \tau\}$ is an antichain.

As $\mathbb{P}_{\mu^+}$ satisfies Clause (3) of Definition 2.3, $\mathbb{P}_{\mu^+}$ has the μ^+-cc. Consequently, there exists a large enough $\beta < \mu^+$ such that

$$B_{r^*} \cup \bigcup \{B_p \mid (\xi, p) \in \tau\} \subseteq \beta.$$

Let $r := r^* \restriction \beta$ and set

$$\sigma := \{(\xi, p \restriction \beta) \mid (\xi, p) \in \tau\}.$$

From the choice of Building Block III, we may find a large enough $\delta < \mu^+$ with $\delta > \beta$ such that $\psi(\delta) = (\beta, r, \sigma)$. As $\beta < \delta$, $r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, the definition of our iteration at step $\delta + 1$ involves appealing to Building Block II with $(\mathbb{P}_\delta, \ell_\delta, c_\delta)$, $r^\star := r * \emptyset_\delta$ and $z := i^\delta_\beta(\sigma)$. For any ordinal $\eta < \mu^+$, denote $G_\eta := \pi_{\mu^+,\eta}[G]$. By the choice of β, and as $\delta > \beta$, we have

$$\tau = \{(\xi, p * \emptyset_{\mu^+}) \mid (\xi, p) \in \sigma\} = \{(\xi, p * \emptyset_{\mu^+}) \mid (\xi, p) \in z\},$$

so that, in $V[G]$,

$$T = \tau_G = \sigma_{G_\beta} = z_{G_\delta}.$$

In addition, $r^* = r^\star * \emptyset_{\mu^+}$.

Finally, as r^* forces τ is a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ, $r^\star$ forces that z is a stationary subset of $(E^\mu_\omega)^V$ that does not reflect in Γ. So, since $\pi_{\mu^+,\delta+1}(r^*) = r^\star * \emptyset_{\delta+1} = \lceil r^\star \rceil^{\mathbb{P}_{\delta+1}}$ is in $G_{\delta+1}$, Clause (e) of Building Block II entails that, in $V[G_{\delta+1}]$, there exists a club in μ which is disjoint from T. In particular, T is nonstationary in $V[G]$, contradicting its very choice. □

Thus, we arrive at the following strengthening of the theorem announced by Sharon in [Sha05].[32]

Corollary 4.26. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, converging to a cardinal κ. Then there exists a forcing extension where the following properties hold:*

(1) κ is a singular strong limit cardinal of countable cofinality;
(2) $2^\kappa = \kappa^{++}$, hence SCH_κ fails;
(3) $\mathrm{Refl}(<\omega, \kappa^+)$ holds.

Proof. Let $\mathbb{L}$ be the inverse limit of the iteration $\langle \mathbb{L}_n; \dot{\mathbb{Q}}_n \mid n < \omega \rangle$, where $\mathbb{L}_0$ is the trivial forcing and for positive integer n, if $\mathbb{1}_{\mathbb{L}_n} \Vdash_{\mathbb{L}_n}$ "κ_{n-1} is supercompact", then $\mathbb{1}_{\mathbb{L}_n} \Vdash_{\mathbb{L}_n}$ "$\dot{\mathbb{Q}}_n$ is a Laver preparation for κ_n above κ_{n-1}". After forcing with $\mathbb{L}$, each κ_n remains supercompact and, moreover, becomes indestructible under κ_n-directed-closed forcing. Also, the cardinals and cofinalities of interest are preserved.

Working in $V^{\mathbb{L}}$, set $\mu := \kappa^+$, $\lambda := \kappa^{++}$ and $\mathbb{C} := \mathrm{Add}(\lambda, 1)$. Finally, work in $W := V^{\mathbb{L} * \dot{\mathbb{C}}}$. Since κ is singular strong limit of cofinality $\omega < \kappa_0$ and κ_0 is supercompact, $2^\kappa = \kappa^+$. Also, thanks to the forcing $\mathbb{C}$, $2^\mu = \mu^+$. Altogether, in W, all the following hold:

- $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-Indestructible supercompact cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;

[32] Recall that by Fact 4.2, the extent of reflection obtained is optimal.

Now, appeal to Theorem 4.25. □

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